Stat 401, section 8.3 The One-Sample *t* Test

notes by Tim Pilachowski

In section 8.2 [9th edition; 8.2a 8th edition], we relied on the Central Limit Theorem for cases where n was "large enough" for the shape of sampling distribution to be close enough to a normal distribution to be able to use the normal distribution table to conduct a hypothesis test about a population mean. [Recall that our rules of thumb are $n > 30$ for situations where σ is known, and $n > 40$ for situations where σ is not known.]

But, what if the sample size n is not large enough?

Starting back in section 5.4, then recalled in section 7.3, we (and the text) noted that, given a population which has a normal distribution, the resulting sampling distributions for any sample size n will retain the symmetry of the population distribution. We will retain this idea as our basic assumption: "The population of interest is normal, so that $X_1, \ldots, X_n$, constitutes a random sample from a normal distribution with both μ and σ unknown."

When n is large enough, the random variable $Z = \dfrac{\overline{X} - \mu}{S/\sqrt{n}}$ has approximately a standard normal distribution.

That is, the size of n helps to ameliorate the effect of having two random variables present in our transformation formula.

When n is small, the additional variability resulting from using S in the denominator means that the sampling distribution will be more spread out than a normal distribution is. As long as the population of interest has a normal, or close to normal, probability distribution, the sampling distribution is modeled by a t distribution with $n - 1$ degrees of freedom. (For theory, equations, and graphs, see Lecture 7.3)

The process for a small-sample hypothesis test of a mean is the same as that for a large-sample situation, with the exception that the test statistic formula is $T = \dfrac{\overline{X} - \mu_0}{S/\sqrt{n}}$, which means that we must specify $\nu = n - 1$ degrees of freedom.

Determination of the P-value is similar: $p = 2 * P(T > t_{\alpha/2,\nu})$ for a two-tailed test, $p = P(T > t_{\alpha,\nu})$ for a one-tailed test. To determine these probabilities in the tail, we'll rely on Table A.8, found in the Appendix of the text. (The decimal accuracy is less than that provided in the z table.)

The decision criteria is the same: If $p \le \alpha$, reject H_0. Otherwise, fail to reject H_0. State the conclusion in the context of the problem.

Example A. Environmental researchers measure the amount of suspended solids in a lake. A sample of size n = 25 had a mean of 59.9 and a standard deviation of 13.27. There is reason to believe that concentration has a normal distribution. a) Test the claim that the mean concentration is different from 54 at a significance level of 0.05. b) Describe what a Type I error would mean in this case. c) Describe what a Type II error would mean in this case.

(Example A continued)

Example B. A car manufacturer claims that a new model gets, on the average, thirty miles per gallon. A consumer group believes the actual mileage is less than 30 mpg and tests 10 cars, obtaining the data below. Does the consumer group have enough statistical evidence to challenge the manufacturer's claim? Conduct a hypothesis test ($\alpha = 0.05$) and state your conclusion. (Assume that distribution of mpg is normal.)

26.6 30.4 32.5 26.3 31.0 25.9 29.7 24.8 30.6 28.1

With a small-sample hypothesis test, it is clear (even more so than it is for a large-sample test) that the test statistic *t* gives us neither the probability that H_0 is true nor the probability of rejecting H_0. A different sample could yield a very different value for the test statistic, resulting in a different conclusion.

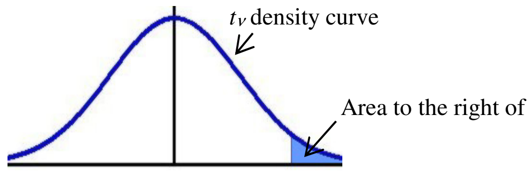
In other words, since the test statistic will vary from one sample to another, it is a random variable. Likewise, the *P*-value is a random variable.

If the null hypothesis is false, we would want a *P*-value close to 0 so that H_0 would be rejected under a wide range of sample sizes and test statistic values.

If the null hypothesis is true, we would want a *P*-value much larger than the significance level so that we would fail to reject H_0 under a wide range of sample sizes and test statistic values.

The text discusses the probability of a Type II error β and determination of sample size. The mathematics involves integrating a "very unpleasant density function", which must be done numerically. "The results are summarized in graphs of β that appear in Appendix Table A.17." We won't be covering this topic in this class.

Appendix Table A.8
t Curve Tail Areas



	ν																	
t	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.1	0.468	0.465	0.463	0.463	0.462	0.462	0.462	0.461	0.461	0.461	0.461	0.461	0.461	0.461	0.461	0.461	0.461	0.461
0.2	0.437	0.430	0.427	0.426	0.425	0.424	0.424	0.423	0.423	0.423	0.423	0.422	0.422	0.422	0.422	0.422	0.422	0.422
0.3	0.407	0.396	0.392	0.390	0.388	0.387	0.386	0.386	0.386	0.385	0.385	0.385	0.384	0.384	0.384	0.384	0.384	0.384
0.4	0.379	0.364	0.358	0.355	0.353	0.352	0.351	0.350	0.349	0.349	0.348	0.348	0.348	0.348	0.347	0.347	0.347	0.347
0.5	0.352	0.333	0.326	0.322	0.319	0.317	0.316	0.315	0.315	0.314	0.313	0.313	0.313	0.312	0.312	0.312	0.312	0.312
0.6	0.328	0.305	0.295	0.290	0.287	0.285	0.284	0.283	0.282	0.281	0.280	0.280	0.279	0.279	0.279	0.278	0.278	0.278
0.7	0.306	0.278	0.267	0.261	0.258	0.255	0.253	0.252	0.251	0.250	0.249	0.249	0.248	0.248	0.247	0.247	0.247	0.246
0.8	0.285	0.254	0.241	0.234	0.230	0.227	0.225	0.223	0.222	0.221	0.220	0.220	0.219	0.219	0.218	0.218	0.217	0.217
0.9	0.267	0.232	0.217	0.210	0.205	0.201	0.199	0.197	0.196	0.195	0.194	0.193	0.192	0.192	0.191	0.191	0.190	0.190
1.0	0.250	0.211	0.196	0.187	0.182	0.178	0.175	0.173	0.172	0.170	0.169	0.169	0.168	0.167	0.167	0.166	0.166	0.165
1.1	0.235	0.193	0.176	0.167	0.161	0.157	0.154	0.152	0.150	0.149	0.147	0.146	0.146	0.145	0.144	0.144	0.143	0.143
1.2	0.221	0.177	0.158	0.148	0.142	0.138	0.135	0.132	0.130	0.129	0.128	0.127	0.126	0.125	0.124	0.124	0.123	0.123
1.3	0.209	0.162	0.142	0.132	0.125	0.121	0.117	0.115	0.113	0.111	0.110	0.109	0.108	0.107	0.107	0.106	0.105	0.105
1.4	0.197	0.148	0.128	0.117	0.110	0.106	0.102	0.100	0.098	0.096	0.095	0.093	0.092	0.092	0.091	0.090	0.090	0.089
1.5	0.187	0.136	0.115	0.104	0.097	0.092	0.089	0.086	0.084	0.082	0.081	0.080	0.079	0.078	0.077	0.077	0.076	0.075
1.6	0.178	0.125	0.104	0.092	0.085	0.080	0.077	0.074	0.072	0.070	0.069	0.068	0.067	0.066	0.065	0.065	0.064	0.064
1.7	0.169	0.116	0.094	0.082	0.075	0.070	0.066	0.064	0.062	0.060	0.059	0.057	0.056	0.056	0.055	0.054	0.054	0.053
1.8	0.161	0.107	0.085	0.073	0.066	0.061	0.057	0.055	0.053	0.051	0.050	0.049	0.048	0.047	0.046	0.045	0.045	0.044
1.9	0.154	0.099	0.077	0.065	0.058	0.053	0.050	0.047	0.045	0.043	0.042	0.041	0.040	0.039	0.038	0.038	0.037	0.037
2.0	0.148	0.092	0.070	0.058	0.051	0.046	0.043	0.040	0.038	0.037	0.035	0.034	0.033	0.033	0.032	0.031	0.031	0.030
2.1	0.141	0.085	0.063	0.052	0.045	0.040	0.037	0.034	0.033	0.031	0.030	0.029	0.028	0.027	0.027	0.026	0.025	0.025
2.2	0.136	0.079	0.058	0.046	0.040	0.035	0.032	0.029	0.028	0.026	0.025	0.024	0.023	0.023	0.022	0.021	0.021	0.021
2.3	0.131	0.074	0.052	0.041	0.035	0.031	0.027	0.025	0.023	0.022	0.021	0.020	0.019	0.019	0.018	0.018	0.017	0.017
2.4	0.126	0.069	0.048	0.037	0.031	0.027	0.024	0.022	0.020	0.019	0.018	0.017	0.016	0.015	0.015	0.014	0.014	0.014
2.5	0.121	0.065	0.044	0.033	0.027	0.023	0.020	0.018	0.017	0.016	0.015	0.014	0.013	0.013	0.012	0.012	0.011	0.011
2.6	0.117	0.061	0.040	0.030	0.024	0.020	0.018	0.016	0.014	0.013	0.012	0.012	0.011	0.010	0.010	0.010	0.009	0.009
2.7	0.113	0.057	0.037	0.027	0.021	0.018	0.015	0.014	0.012	0.011	0.010	0.010	0.009	0.009	0.008	0.008	0.008	0.007
2.8	0.109	0.054	0.034	0.024	0.019	0.016	0.013	0.012	0.010	0.009	0.009	0.008	0.008	0.007	0.007	0.006	0.006	0.006
2.9	0.106	0.051	0.031	0.022	0.017	0.014	0.011	0.010	0.009	0.008	0.007	0.007	0.006	0.006	0.005	0.005	0.005	0.005
3.0	0.102	0.048	0.029	0.020	0.015	0.012	0.010	0.009	0.007	0.007	0.006	0.006	0.005	0.005	0.004	0.004	0.004	0.004
3.1	0.099	0.045	0.027	0.018	0.013	0.011	0.009	0.007	0.006	0.006	0.005	0.005	0.004	0.004	0.004	0.003	0.003	0.003
3.2	0.096	0.043	0.025	0.016	0.012	0.009	0.008	0.006	0.005	0.005	0.004	0.004	0.003	0.003	0.003	0.003	0.003	0.002
3.3	0.094	0.040	0.023	0.015	0.011	0.008	0.007	0.005	0.005	0.004	0.004	0.003	0.003	0.003	0.002	0.002	0.002	0.002
3.4	0.091	0.038	0.021	0.014	0.010	0.007	0.006	0.005	0.004	0.003	0.003	0.003	0.002	0.002	0.002	0.002	0.002	0.002
3.5	0.089	0.036	0.020	0.012	0.009	0.006	0.005	0.004	0.003	0.003	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.001
3.6	0.086	0.035	0.018	0.011	0.008	0.006	0.004	0.003	0.003	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.001	0.001
3.7	0.084	0.033	0.017	0.010	0.007	0.005	0.004	0.003	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001
3.8	0.082	0.031	0.016	0.010	0.006	0.004	0.003	0.003	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001
3.9	0.080	0.030	0.015	0.009	0.006	0.004	0.003	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001

4.0	0.078	0.029	0.014	0.008	0.005	0.004	0.003	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000

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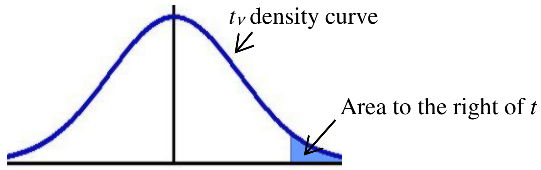

	ν																
t	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**35**	**40**	**60**	**120**	**∞**
0.0	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500	0.500
0.1	0.461	0.461	0.461	0.461	0.461	0.461	0.461	0.461	0.461	0.461	0.461	0.461	0.460	0.460	0.460	0.460	0.460
0.2	0.422	0.422	0.422	0.422	0.422	0.422	0.422	0.422	0.421	0.421	0.421	0.421	0.421	0.421	0.421	0.421	0.421
0.3	0.384	0.384	0.384	0.383	0.383	0.383	0.383	0.383	0.383	0.383	0.383	0.383	0.383	0.383	0.383	0.382	0.382
0.4	0.347	0.347	0.347	0.347	0.346	0.346	0.346	0.346	0.346	0.346	0.346	0.346	0.346	0.346	0.345	0.345	0.345
0.5	0.311	0.311	0.311	0.311	0.311	0.311	0.311	0.311	0.311	0.310	0.310	0.310	0.310	0.310	0.309	0.309	0.309
0.6	0.278	0.278	0.277	0.277	0.277	0.277	0.277	0.277	0.277	0.277	0.277	0.277	0.276	0.276	0.275	0.275	0.274
0.7	0.246	0.246	0.246	0.246	0.245	0.245	0.245	0.245	0.245	0.245	0.245	0.245	0.244	0.244	0.243	0.243	0.242
0.8	0.217	0.217	0.216	0.216	0.216	0.216	0.216	0.215	0.215	0.215	0.215	0.215	0.215	0.214	0.213	0.213	0.212
0.9	0.190	0.189	0.189	0.189	0.189	0.189	0.188	0.188	0.188	0.188	0.188	0.188	0.187	0.187	0.186	0.185	0.184
1.0	0.165	0.165	0.164	0.164	0.164	0.164	0.163	0.163	0.163	0.163	0.163	0.163	0.162	0.162	0.161	0.160	0.159
1.1	0.143	0.142	0.142	0.142	0.141	0.141	0.141	0.141	0.141	0.140	0.140	0.140	0.139	0.139	0.138	0.137	0.136
1.2	0.122	0.122	0.122	0.121	0.121	0.121	0.121	0.120	0.120	0.120	0.120	0.120	0.119	0.119	0.117	0.116	0.115
1.3	0.105	0.104	0.104	0.104	0.103	0.103	0.103	0.103	0.102	0.102	0.102	0.102	0.101	0.101	0.099	0.098	0.097
1.4	0.089	0.088	0.088	0.088	0.087	0.087	0.087	0.087	0.086	0.086	0.086	0.086	0.085	0.085	0.083	0.082	0.081
1.5	0.075	0.075	0.074	0.074	0.074	0.073	0.073	0.073	0.073	0.072	0.072	0.072	0.071	0.071	0.069	0.068	0.067
1.6	0.063	0.063	0.062	0.062	0.062	0.061	0.061	0.061	0.061	0.060	0.060	0.060	0.059	0.059	0.057	0.056	0.055
1.7	0.053	0.052	0.052	0.052	0.051	0.051	0.051	0.051	0.050	0.050	0.050	0.050	0.049	0.048	0.047	0.046	0.045
1.8	0.044	0.043	0.043	0.043	0.042	0.042	0.042	0.042	0.042	0.041	0.041	0.041	0.040	0.040	0.038	0.037	0.036
1.9	0.036	0.036	0.036	0.035	0.035	0.035	0.035	0.034	0.034	0.034	0.034	0.034	0.033	0.032	0.031	0.030	0.029
2.0	0.030	0.030	0.029	0.029	0.029	0.028	0.028	0.028	0.028	0.028	0.027	0.027	0.027	0.026	0.025	0.024	0.023
2.1	0.025	0.024	0.024	0.024	0.023	0.023	0.023	0.023	0.023	0.022	0.022	0.022	0.022	0.021	0.020	0.019	0.018
2.2	0.020	0.020	0.020	0.019	0.019	0.019	0.019	0.018	0.018	0.018	0.018	0.018	0.017	0.017	0.016	0.015	0.014
2.3	0.016	0.016	0.016	0.016	0.015	0.015	0.015	0.015	0.015	0.015	0.014	0.014	0.014	0.013	0.012	0.012	0.011
2.4	0.013	0.013	0.013	0.013	0.012	0.012	0.012	0.012	0.012	0.012	0.012	0.011	0.011	0.011	0.010	0.009	0.008
2.5	0.011	0.011	0.010	0.010	0.010	0.010	0.010	0.010	0.009	0.009	0.009	0.009	0.009	0.008	0.008	0.007	0.006
2.6	0.009	0.009	0.008	0.008	0.008	0.008	0.008	0.008	0.007	0.007	0.007	0.007	0.007	0.006	0.006	0.005	0.005
2.7	0.007	0.007	0.007	0.007	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.006	0.005	0.005	0.004	0.004	0.003
2.8	0.006	0.006	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.005	0.004	0.004	0.004	0.004	0.003	0.003	0.003
2.9	0.005	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.004	0.003	0.003	0.003	0.003	0.002	0.002
3.0	0.004	0.004	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.003	0.002	0.002	0.002	0.002	0.001
3.1	0.003	0.003	0.003	0.003	0.003	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.001
3.2	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.001	0.001
3.3	0.002	0.002	0.002	0.002	0.002	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000
3.4	0.002	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000
3.5	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000
3.6	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000
3.7	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3.8	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
3.9	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

4.0	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000		0.000