Stat 401, section 8.4 Tests Concerning a Population Proportion

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Do you remember binomial distributions and proportions? (See Stat 400 Lecture 6.1c.) Section 8.4 [8th edition 8.3] addresses hypothesis tests used for investigating population **proportions** (as opposed to means).

Vocabulary:

The point estimate for a population proportion will be a sample proportion $\hat{p} = \dfrac{\text{number of successes}}{\text{total number tested}} = \dfrac{X}{n}$.

Remember the z-score formula for proportions? We'll do a little mathematical finagling with the the z-score formula for binomial distributions.

$$Z = \frac{X - np}{\sqrt{npq}} = \frac{\frac{X}{n} - \frac{np}{n}}{\frac{\sqrt{npq}}{n}} = \frac{\frac{X}{n} - \frac{np}{n}}{\sqrt{\frac{npq}{n^2}}} = \frac{\hat{p} - p}{\sqrt{pq/n}}$$

For the population proportion p in the numerator, when we do a hypothesis test, we'll use the value from the null hypothesis, p_0. Likewise, for the standard error (in the denominator), since we're assuming the population parameter is p_0, we'll use the null value p_0, then $q_0 = 1 - p_0$.

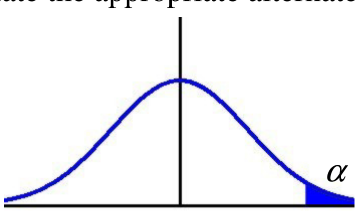
Example A – sample proportion: A pharmaceutical company has developed a new drug, which a research hospital tests by treating 50 patients. Of those tested, 28 report positive results. What is the sample proportion?

point estimate of p =

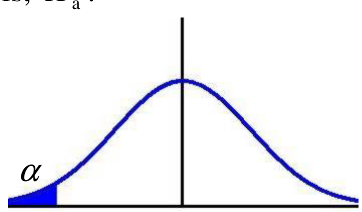
For the hypothesis test involving proportions, we'll use the same rule of thumb as we did for using a normal distribution to approximate a binomial distribution. If both $n * p_0 \geq 10$ and also $n * q_0 \geq 10$, then the hypothesis test is considered a large-sample test.

Here are the steps as outlined in the text.

1. Identify the population parameter under scrutiny and describe it in the context of the described situation.
2. Determine the null value and state the null hypothesis, H_0.
3. State the appropriate alternate hypothesis, H_a.



$H_0: p \leq p_0$
$H_a: p > p_0$



$H_0: p \geq p_0$
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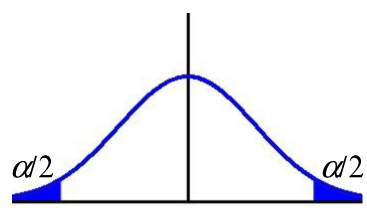

$H_0: p = p_0$
$H_a: p \neq p_0$

4. Conduct the test to verify a large-sample situation, and give the formula for the computed value of the test statistic: $Z = \dfrac{\hat{p} - p_0}{\sqrt{\frac{p_0 q_0}{n}}}$.
5. Compute any necessary sample statistics, and use them to compute the value of the test statistic.
6. Determine the P-value: $p = 2\left(1 - \Phi\left(\left| z \right|\right)\right)$ for a two-tailed test, $p = 1 - \Phi\left(\left| z \right|\right) = \Phi\left(-\left| z \right|\right)$ for a one-tailed test.
7. If $p \leq \alpha$, reject H_0. Otherwise, fail to reject H_0. State the conclusion in the context of the problem.

Example A – hypothesis test: A pharmaceutical company says a new drug is more than 70% successful. A research hospital tests this by treating 50 patients. Of those treated, 28 report positive results. Is there sufficient evidence to challenge the company's claim? ($\alpha = 0.05$)

When the given situation doesn't meet the large-sample rule of thumb (i.e. we have either $n * p_0 < 10$ or $n * q_0 < 10$), hypothesis tests involving proportions will be based directly on the binomial distribution rather than on a normal approximation. In a small-sample case, the determination of the P-value changes from the formula used above. For example, for a one-sided upper bound test, $H_a : p > p_0$,

notes on the derivation:

$$
\begin{aligned}
P\text{-value} &= P(X \geq x \text{ when } H_0 \text{ is true}) \\
&= P(X \geq x \text{ when } X \sim \text{Bin}(n, p_0)) \\
&= 1 - P(X \leq x - 1 \text{ when } X \sim \text{Bin}(n, p_0)) \\
&= 1 - B(x - 1; n, p_0)
\end{aligned}
$$

For a one-sided lower bound test, $H_a : p < p_0$, P-value = $B(x; n, p_0)$.
For a two-sided test, the P-value will equal 2 times the smaller of $1 - B(x - 1; n, p_0)$ and $B(x; n, p_0)$.

Because a binomial distribution is discrete, if we use the binomial cumulative distribution tables, we may not be able to compare the P-value to exactly the desired significance level. Instead, we'll compare to the closest achievable level found in the table. It will be preferable to use a calculator/computer utility or software package such as Minitab to calculate the P-value.

Example B (text ch 8 supplementary exercise #80). The National Institute for Standards and Technology (NIST) says that at most 2 out of 100 purchases at a store should have incorrectly scanned prices. In a 2005 article, researchers reported scanners at a Wal-Mart coming up with the wrong price 8.5% of the time. Suppose this was based on 200 purchases. Carry out a test (level of significance 0.05) to decide whether or not the NIST benchmark is satisfied.

Example C. Suppose that quality control standards require that no more than 5% of manufactured components are defective. In a sample of 25, how many observed defective components would result in questioning the quality of a production run ($\alpha = 0.05$)?

As with the small-sample hypothesis test of a mean, we won't be covering the probability of a Type II error β and determination of sample size in this class.