Stat 401, section 10.2 Multiple Comparisons in ANOVA

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In section 10.1, we compared more than two populations using a single-factor analysis of variance (ANOVA). In Example A, our test statistic f led to a conclusion that the methods used to teach three different groups of sixth graders had no statistically significant effect on the students' scores ($\alpha = 0.05$) on the *Mathematics Anxiety Scale for Children* (MASC). In cases such as this, analysis ends, because no differences among the μ_i's have been identified.

If we *had* found differences among the three methods, we would have wanted to do pairwise comparisons to identify whether there were differences between μ_1 and μ_2, between μ_2 and μ_3, and/or between μ_1 and μ_3.

A method for carrying out this further analysis is called a **multiple comparisons procedure.**

Several of the most frequently used procedures are based on calculating confidence intervals for each pairwise difference $\mu_i - \mu_j$ for each $i < j$ pair.

For Example A's $I = 3$, we would have had $\binom{3}{2} = \frac{3!}{2!1!} = 3$ CIs. For $I = 4$, the number would increase to $\binom{4}{2} = \frac{4!}{2!2!} = 6$ CIs. For $I = 5$, the number would increase to $\binom{5}{2} = \frac{5!}{2!3!} = 10$ CIs.

In general, for number of populations/treatments I, we would need $\binom{I}{2} = \frac{I!}{2!(I-2)!} = \frac{I(I-1)}{2}$ CIs.

Then, if the interval for $\mu_1 - \mu_2$ does not include 0, conclude that μ_1 and μ_2 *differ significantly* from one another; if the interval does include 0, the two μ's are judged not significantly different. Following the same line of reasoning for each of the other intervals, we can discern for each pair of μ's whether or not they differ significantly from one another.

This text focuses on **Tukey's Procedure** (the T method) which controls the *simultaneous* confidence level for all of the $\frac{I(I-1)}{2}$ intervals. Tukey's procedure involves the use of a probability distribution called the **Studentized range distribution**, which depends on two parameters: a numerator degrees of freedom $m = I$ and a denominator degrees of freedom $\nu = I(J-1)$. The upper-tail α critical value (i.e. the critical value that places probability a in the upper tail) is denoted $Q_{\alpha,m,\nu}$. Values of $Q_{\alpha,m,\nu}$ are given in Appendix Table A.10.

Note that numerator degrees of freedom $m = I$, and <u>not</u> $I - 1$ as in the section 10.1 F test.

Proposition:

With probability $1 - \alpha$, $\left(\overline{X}_{i,\cdot} - \overline{X}_{j,\cdot}\right) - Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}} \le \mu_i - \mu_j \le \left(\overline{X}_{i,\cdot} - \overline{X}_{j,\cdot}\right) + Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}}$, for every $i = 1, 2, \ldots, I < j = 1, 2, \ldots, J$.

When the computed $\overline{x}_{i,\cdot}$, $\overline{x}_{j,\cdot}$, and MSE are substituted into the formula above, the result is a collection of confidence intervals with *simultaneous* confidence level $100(1 - \alpha)\%$ for all pairwise differences $\mu_i - \mu_j$ where $i < j$. Each interval that does not include 0 yields the conclusion that the corresponding values of μ_i and μ_j differ significantly from one another.

Two notes:

We won't be proving how the generated confidence intervals have *simultaneous* confidence level $100(1 - \alpha)\%$. Read through the text's explanation "The Interpretation of α in Tukey's Method."

Since we are not really interested in the lower and upper limits of the CIs but only in which include 0 and which do not, we can use a simpler algorithm which avoids much of the arithmetic associated with the formula above.

The *T* Method for Identifying Significantly Different μ_i's

Select α, extract $Q_{\alpha,I,I(J-1)}$ from Appendix Table A.10, and calculate $w = Q_{\alpha,I,I(J-1)}\sqrt{\frac{MSE}{J}}$.

List the sample means in increasing order, and underline those pairs that differ by less than w.
Any pair of sample means <u>not</u> underscored by the same line corresponds to a pair of population or treatment means that are judged significantly different.

$\bar{x}_{a,\cdot} < \bar{x}_{b,\cdot} < \bar{x}_{c,\cdot} < \bar{x}_{d,\cdot} < \bar{x}_{e,\cdot}$

$\bar{x}_{a,\cdot} < \bar{x}_{b,\cdot} < \bar{x}_{c,\cdot} < \bar{x}_{d,\cdot} < \bar{x}_{e,\cdot}$

Example A. A paper in *Measurement and Evaluation in Counseling and Development* (Oct 90, pp. 121–127) discussed a survey instrument called the *Mathematics Anxiety Scale for Children* (MASC). Suppose the MASC was administered to six groups of five sixth graders, with each group having been taught using a different method. Test whether the results of the three methods differ ($\alpha = 0.05$). Data are as follows.

Group 1	67	50	70	60	55	$x_{1.} =$	$\bar{x}_{1.} =$
Group 2	49	32	65	39	43	$x_{2.} =$	$\bar{x}_{2.} =$
Group 3	40	39	41	60	45	$x_{3.} =$	$\bar{x}_{3.} =$
Group 4	75	70	70	75	70	$x_{4.} =$	$\bar{x}_{4.} =$
Group 5	28	33	34	30	29	$x_{5.} =$	$\bar{x}_{5.} =$
Group 6	28	35	34	29	33	$x_{6.} =$	$\bar{x}_{6.} =$
						$x_{..} =$	$\bar{x}_{..} =$

Source of Variation	**Degrees of Freedom**	**Sum of Squares**	**Mean Square**	***f***
Treatments				
Error				X X
Total			X X	X X

critical value c (from Table A.9) =

conclusion:

$I =$ $J =$ $m =$ $\nu =$ $Q_{\alpha,I,I(J-1)} =$ $w =$

conclusion:

Appendix Table A.9 Critical Values for *F* Distributions

		ν_1 = numerator df								
ν_2	α	1	2	3	4	5	6	7	8	9
13	0.100	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16
	0.050	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
	0.010	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19
	0.001	17.82	12.31	10.21	9.07	8.35	7.86	7.49	7.21	6.98
14	0.100	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12
	0.050	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
	0.010	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03
	0.001	17.14	11.78	9.73	8.62	7.92	7.44	7.08	6.80	6.58
15	0.100	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09
	0.050	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59
	0.010	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89
	0.001	16.59	11.34	9.34	8.25	7.57	7.09	6.74	6.47	6.26
16	0.100	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06
	0.050	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54
	0.010	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78
	0.001	16.12	10.97	9.01	7.94	7.27	6.80	6.46	6.19	5.98
17	0.100	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03
	0.050	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49
	0.010	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68
	0.001	15.72	10.66	8.73	7.68	7.02	6.56	6.22	5.96	5.75
18	0.100	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00
	0.050	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46
	0.010	8.29	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60
	0.001	15.38	10.39	8.49	7.46	6.81	6.35	6.02	5.76	5.56
19	0.100	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98
	0.050	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42
	0.010	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52
	0.001	15.08	10.16	8.28	7.27	6.62	6.18	5.85	5.59	5.39
20	0.100	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96
	0.050	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39
	0.010	8.10	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46
	0.001	14.82	9.95	8.10	7.10	6.46	6.02	5.69	5.44	5.24
21	0.100	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95
	0.050	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37
	0.010	8.02	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40
	0.001	14.59	9.77	7.94	6.95	6.32	5.88	5.56	5.31	5.11
22	0.100	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93
	0.050	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34
	0.010	7.95	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35
	0.001	14.38	9.61	7.80	6.81	6.19	5.76	5.44	5.19	4.99
23	0.100	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92
	0.050	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32
	0.010	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30
	0.001	14.20	9.47	7.67	6.70	6.08	5.65	5.33	5.09	4.89
24	0.100	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91
	0.050	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30
	0.010	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26
	0.001	14.03	9.34	7.55	6.59	5.98	5.55	5.23	4.99	4.80

Appendix Table A.9 Critical Values for *F* Distributions

		ν_1 = numerator df								
ν_2	α	10	12	15	20	25	30	40	50	60
	0.100	2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.92	1.90
13	0.050	2.67	2.60	2.53	2.46	2.41	2.38	2.34	2.31	2.30
	0.010	4.10	3.96	3.82	3.66	3.57	3.51	3.43	3.38	3.34
	0.001	6.80	6.52	6.23	5.93	5.75	5.63	5.47	5.37	5.30
	0.100	2.10	2.05	2.01	1.96	1.93	1.91	1.89	1.87	1.86
14	0.050	2.60	2.53	2.46	2.39	2.34	2.31	2.27	2.24	2.22
	0.010	3.94	3.80	3.66	3.51	3.41	3.35	3.27	3.22	3.18
	0.001	6.40	6.13	5.85	5.56	5.38	5.25	5.10	5.00	4.94
	0.100	2.06	2.02	1.97	1.92	1.89	1.87	1.85	1.83	1.82
15	0.050	2.54	2.48	2.40	2.33	2.28	2.25	2.20	2.18	2.16
	0.010	3.80	3.67	3.52	3.37	3.28	3.21	3.13	3.08	3.05
	0.001	6.08	5.81	5.54	5.25	5.07	4.95	4.80	4.70	4.64
	0.100	2.03	1.99	1.94	1.89	1.86	1.84	1.81	1.79	1.78
16	0.050	2.49	2.42	2.35	2.28	2.23	2.19	2.15	2.12	2.11
	0.010	3.69	3.55	3.41	3.26	3.16	3.10	3.02	2.97	2.93
	0.001	5.81	5.55	5.27	4.99	4.82	4.70	4.54	4.45	4.39
	0.100	2.00	1.96	1.91	1.86	1.83	1.81	1.78	1.76	1.75
17	0.050	2.45	2.38	2.31	2.23	2.18	2.15	2.10	2.08	2.06
	0.010	3.59	3.46	3.31	3.16	3.07	3.00	2.92	2.87	2.83
	0.001	5.58	5.32	5.05	4.78	4.60	4.48	4.33	4.24	4.18
	0.100	1.98	1.93	1.89	1.84	1.80	1.78	1.75	1.74	1.72
18	0.050	2.41	2.34	2.27	2.19	2.14	2.11	2.06	2.04	2.02
	0.010	3.51	3.37	3.23	3.08	2.98	2.92	2.84	2.78	2.75
	0.001	5.39	5.13	4.87	4.59	4.42	4.30	4.15	4.06	4.00
	0.100	1.96	1.91	1.86	1.81	1.78	1.76	1.73	1.71	1.70
19	0.050	2.38	2.31	2.23	2.16	2.11	2.07	2.03	2.00	1.98
	0.010	3.43	3.30	3.15	3.00	2.91	2.84	2.76	2.71	2.67
	0.001	5.22	4.97	4.70	4.43	4.26	4.14	3.99	3.90	3.84
	0.100	1.94	1.89	1.84	1.79	1.76	1.74	1.71	1.69	1.68
20	0.050	2.35	2.28	2.20	2.12	2.07	2.04	1.99	1.97	1.95
	0.010	3.37	3.23	3.09	2.94	2.84	2.78	2.69	2.64	2.61
	0.001	5.08	4.82	4.56	4.29	4.12	4.00	3.86	3.77	3.70
	0.100	1.92	1.87	1.83	1.78	1.74	1.72	1.69	1.67	1.66
21	0.050	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.94	1.92
	0.010	3.31	3.17	3.03	2.88	2.79	2.72	2.64	2.58	2.55
	0.001	4.95	4.70	4.44	4.17	4.00	3.88	3.74	3.64	3.58
	0.100	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.65	1.64
22	0.050	2.30	2.23	2.15	2.07	2.02	1.98	1.94	1.91	1.89
	0.010	3.26	3.12	2.98	2.83	2.73	2.67	2.58	2.53	2.50
	0.001	4.83	4.58	4.33	4.06	3.89	3.78	3.63	3.54	3.48
	0.100	1.89	1.84	1.80	1.74	1.71	1.69	1.66	1.64	1.62
23	0.050	2.27	2.20	2.13	2.05	2.00	1.96	1.91	1.88	1.86
	0.010	3.21	3.07	2.93	2.78	2.69	2.62	2.54	2.48	2.45
	0.001	4.73	4.48	4.23	3.96	3.79	3.68	3.53	3.44	3.38
	0.100	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.62	1.61
24	0.050	2.25	2.18	2.11	2.03	1.97	1.94	1.89	1.86	1.84
	0.010	3.17	3.03	2.89	2.74	2.64	2.58	2.49	2.44	2.40
	0.001	4.64	4.39	4.14	3.87	3.71	3.59	3.45	3.36	3.29

Appendix Table A.10 Critical Values for Studentized Range Distributions

		m										
ν	α	2	3	4	5	6	7	8	9	10	11	12
5	0.05	3.64	4.60	5.22	5.67	6.03	6.33	6.58	6.80	6.99	7.17	7.32
	0.01	5.70	6.98	7.80	8.42	8.91	9.32	9.67	9.97	10.24	10.48	10.70
6	0.05	3.46	4.34	4.90	5.30	5.63	5.90	6.12	6.32	6.49	6.65	6.79
	0.01	5.24	6.33	7.03	7.56	7.97	8.32	8.61	8.87	9.10	9.30	9.48
7	0.05	3.34	4.16	4.68	5.06	5.36	5.61	5.82	6.00	6.16	6.30	6.43
	0.01	4.95	5.92	6.54	7.01	7.37	7.68	7.94	8.17	8.37	8.55	8.71
8	0.05	3.26	4.04	4.53	4.89	5.17	5.40	5.60	5.77	5.92	6.05	6.18
	0.01	4.75	5.64	6.20	6.62	6.96	7.24	7.47	7.68	7.86	8.03	8.18
9	0.05	3.20	3.95	4.41	4.76	5.02	5.24	5.43	5.59	5.74	5.87	5.98
	0.01	4.60	5.43	5.96	6.35	6.66	6.91	7.13	7.33	7.49	7.65	7.78
10	0.05	3.15	3.88	4.33	4.65	4.91	5.12	5.30	5.46	5.60	5.72	5.83
	0.01	4.48	5.27	5.77	6.14	6.43	6.67	6.87	7.05	7.21	7.36	7.49
11	0.05	3.11	3.82	4.26	4.57	4.82	5.03	5.20	5.35	5.34	5.61	5.71
	0.01	4.39	5.15	5.62	5.97	6.25	6.48	6.67	6.84	6.99	7.13	7.25
12	0.05	3.08	3.77	4.20	4.51	4.75	4.95	5.12	5.27	5.39	5.51	5.61
	0.01	4.32	5.05	5.50	5.84	6.10	6.32	6.51	6.67	6.81	6.94	7.06
13	0.05	3.06	3.73	4.15	4.45	4.69	4.88	5.05	5.19	5.32	5.43	5.53
	0.01	4.26	4.96	5.40	5.73	5.98	6.19	6.37	6.53	6.67	6.79	6.90
14	0.05	3.03	3.70	4.11	4.41	4.64	4.83	4.99	5.13	5.25	5.36	5.46
	0.01	4.21	4.89	5.32	5.63	5.88	6.08	6.26	6.41	6.54	6.66	6.77
15	0.05	3.01	3.67	4.08	4.37	4.59	4.78	4.94	5.08	5.20	5.31	5.40
	0.01	4.17	4.84	5.25	5.56	5.80	5.99	6.16	6.31	6.44	6.55	6.66
16	0.05	3.00	3.65	4.05	4.33	4.56	4.74	4.90	5.03	5.15	5.26	5.35
	0.01	4.13	4.79	5.19	5.49	5.72	5.92	6.08	6.22	6.35	6.46	6.56
17	0.05	2.98	3.63	4.02	4.30	4.52	4.70	4.86	4.99	5.11	5.21	5.31
	0.01	4.10	4.74	5.14	5.43	5.66	5.85	6.01	6.15	6.27	6.38	6.48
18	0.05	2.97	3.61	4.00	4.28	4.49	4.67	4.82	4.96	5.07	5.17	5.27
	0.01	4.07	4.70	5.09	5.38	5.60	5.79	5.94	6.08	6.20	6.31	6.41
19	0.05	2.96	3.59	3.98	4.25	4.47	4.65	4.79	4.92	5.04	5.14	5.23
	0.01	4.05	4.67	5.05	5.33	5.55	5.73	5.89	6.02	6.14	6.25	6.34
20	0.05	2.95	3.59	3.96	4.23	4.45	4.62	4.77	4.90	5.01	5.11	5.20
	0.01	4.02	4.64	5.02	5.29	5.51	5.69	5.84	5.97	6.09	6.19	6.28
24	0.05	2.92	3.53	3.90	4.17	4.37	4.54	4.68	4.81	4.92	5.01	5.10
	0.01	3.96	4.55	4.91	5.17	5.37	5.54	5.69	5.81	5.92	6.02	6.11
30	0.05	2.89	3.49	3.85	4.10	4.30	4.46	4.60	4.72	4.82	4.92	5.00
	0.01	3.89	4.45	4.80	5.05	5.24	5.40	5.54	5.65	5.76	5.85	5.93
40	0.05	2.86	3.44	3.79	4.04	4.23	4.39	4.52	4.63	4.73	4.82	4.90
	0.01	3.82	4.37	4.70	4.93	5.11	5.26	5.39	5.50	5.60	5.69	5.76
60	0.05	2.83	3.40	3.74	3.98	4.16	4.36	4.44	4.55	4.65	4.73	4.81
	0.01	3.76	4.28	4.59	4.82	4.99	5.13	5.25	5.36	5.45	5.53	5.60
120	0.05	2.80	3.36	3.68	3.92	4.10	4.24	4.36	4.47	4.56	4.64	4.71
	0.01	3.70	4.20	4.50	4.71	4.87	5.01	5.12	5.21	5.30	5.37	5.44
∞	0.05	2.77	3.30	3.63	3.86	4.03	4.17	4.29	4.39	4.47	4.55	4.62
	0.01	3.64	4.12	4.40	4.60	4.76	4.88	4.99	5.08	5.16	5.23	5.29