Stat 401, section 10.3 More on Single Factor ANOVA

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First the good news: We're going to skip past "The ANOVA Model" and "β for the F Test" as being beyond the scope of what is needed for Stat 401.

For sections 10.1 and 10.2, we had all sample sizes equal, i.e. $n_1 = n_2 = \ldots = n_I = J$. But what if all sample sizes are not equal? We can still conduct ANOVA, but with some adjustments to formulas (see below).

When the number of treatments or populations is $I = 2$, either a two-sample t test or ANOVA can be used. However, the two-sample t test is more flexible than the F test. First, it is valid without the assumption of equal population variances. Second, it can be used for either a two-tailed test $\left(H_{\mathrm{a}} : \mu_1 = \mu_2\right)$ or a one-tailed test $\left(H_{\mathrm{a}} : \mu_1 < \mu_2, \quad H_{\mathrm{a}} : \mu_1 > \mu_2\right)$.

When the number of treatments or populations is $I \geq 3$, there is unfortunately no general test procedure known to have good properties without assuming equal variances.

Let $J_1,\ J_2,\ \ldots,\ J_I$ represent the I sample sizes, and let $J_1 + J_2 + \ldots + J_I = n$ total observations.

Individual sample means will be denoted by random variables $\overline{X}_{1\cdot}, \overline{X}_{2\cdot}, \ldots, \overline{X}_{I\cdot}$, where $\overline{X}_{i\cdot} = \dfrac{\sum_{j=1}^{J_i} X_{i,j}}{J_i}$.

Note: We'll still use the notation $x_{i\cdot} = \sum_{j=1}^{J_i} x_{i,j}$, so that calculated $\overline{x}_{i\cdot} = \dfrac{x_{i\cdot}}{J_i}$.

The average of all $J_1 + J_2 + \ldots + J_I$ observations, the grand mean**,** is given by $\overline{X}_{\cdot\cdot} = \dfrac{\sum_{i=1}^{I}\sum_{j=1}^{J_i} X_{i,j}}{n}$.

Note: We'll still use the notation $x_{\cdot\cdot} = \sum_{i=1}^{I}\sum_{j=1}^{J_i} x_{i,j}$, so that calculated $\overline{x}_{\cdot\cdot} = \dfrac{x_{\cdot\cdot}}{n}$.

The hypotheses will be

$$H_0 : \mu_1 = \mu_2 = \ldots = \mu_I \text{ versus}$$
$$H_{\mathrm{a}} : \text{ at least two the of the } \mu_i\text{'s are different.}$$

We need a random sample from each population or treatment.

Basic assumptions remain the same.
The $x_{i,j}$'s within any particular sample are independent (i.e. we have a random sample taken from the ith population or treatment distribution).
Different samples are independent of one another.
Each of the I population or treatment distributions is normal, and each has the same variance σ^2.

Preliminary calculations – sums of squares:

The **total sum of squares** (SST) $= \sum_{i=1}^{I}\sum_{j=1}^{J_i} \left(x_{i,j} - \overline{x}_{\cdot\cdot}\right)^2 = \sum_{i=1}^{I}\sum_{j=1}^{J_i} \left(x_{i,j}\right)^2 - \frac{1}{n}\left(x_{\cdot\cdot}\right)^2, \quad \text{df} = n-1$.

The **treatment sum of squares** (SSTr) $= \sum_{i=1}^{I}\sum_{j=1}^{J_i} \left(\overline{x}_{i\cdot} - \overline{x}_{\cdot\cdot}\right)^2 = \sum_{i=1}^{I} \frac{1}{J_i}\left(x_{i\cdot}\right)^2 - \frac{1}{n}\left(x_{\cdot\cdot}\right)^2, \quad \text{df} = I-1$.

The **error sum of squares** (SSE) $= \sum_{i=1}^{I}\sum_{j=1}^{J_i} \left(x_{i,j} - \overline{x}_{i\cdot}\right)^2 = \text{SST} - \text{SSTr}, \quad \text{df} = \sum_{i=1}^{I} \left(J_i - 1\right) = n - I$.

The **mean square for treatments** (MSTr) $= \dfrac{\text{SSTr}}{I-1}$. The **mean square for error** (MSE) $= \dfrac{\text{SSE}}{n-I}$.

The test statistic remains the ratio of the two mean squares, $F = \dfrac{\text{MSTr}}{\text{MSE}}$.

When we have statistical software we will use it to calculate the *P*-value. When calculating by hand, we'll use Appendix Table A.9 to determine the critical value *c* for $\alpha = 0.10, 0.05, 0.01$, and 0.001. A calculated test statistic $f \geq c$ implies $p \leq \alpha$, in which case we will reject the null hypothesis.

When we reject the null hypothesis, a multiple comparison procedure is needed. This text uses a variation of Tukey's Procedure (the *T* method) for use when the sample sizes are "reasonably close". In this approach, the value of each $w_{i,j}$ depends upon the sizes of the two samples being compared: $w = Q_{\alpha,I,n-I}\sqrt{\dfrac{MSE}{2}\left(\dfrac{1}{J_i}+\dfrac{1}{J_j}\right)}$.

As before, list the sample means in increasing order, and underline those pairs that differ by less than $w_{i,j}$.

Any pair of sample means <u>*not*</u> underscored by the same line corresponds to a pair of population or treatment means that are judged significantly different.

Example A. A paper in *Measurement and Evaluation in Counseling and Development* (Oct 90, pp. 121–127) discussed a survey instrument called the *Mathematics Anxiety Scale for Children* (MASC). Suppose the MASC was administered to three groups of sixth graders, with each group having been taught using a different method. Test whether the results of the three methods differ ($\alpha = 0.05$). Data is as follows.

Group 1	67	50	70	60	55	75		$J_1 =$	$x_{1.} =$	$\bar{x}_{1.} =$
Group 2	49	32	65	39	43			$J_2 =$	$x_{2.} =$	$\bar{x}_{2.} =$
Group 3	40	39	41	60	45	30	28	$J_3 =$	$x_{3.} =$	$\bar{x}_{3.} =$
								$n =$	$x_{..} =$	$\bar{x}_{..} =$

hypotheses:

$I =$ df numerator = df denominator =

$\sum\sum (x_{i,j})^2 =$

SST =

SSTr =

SSE =

MSTr =

MSE =

$f =$

Source of Variation	Degrees of Freedom	Sum of Squares	Mean Square	*f*
Treatments				
Error				X X
Total			X X	X X

critical value *c* (from Table A.9) =

ANOVA conclusion:

T-method:

$I =$ $n - I =$ $Q_{\alpha, I, n-I} =$

T-method conclusion: