MATH 462
PROBLEM SET 2

1) I want you to find the Fourier cosine series for $f(x) = e^x$ on the interval $(0, l)$. Don't actually integrate and calculate the coefficients, this is very tedious. Instead proceed as follows. If the function $f(x)$ is reasonably "nice" (note that e^x is C^∞ and in fact, analytic which seems pretty nice), then to obtain a Fourier series for $f'(x)$ we MAY simply differentiate the Fourier series for f term by term.
a) Use this fact to find the Fourier cosine series for $f(x) = e^x$ on the interval $(0, l)$. You will need to differentiate twice to get back to cosines.
b) You got zero as the answer. Well Done!!! What happened?
c) Correct the mistake and find the real answer.

2) (Energy and Uniqueness to Finite End String Problem with Fixed Ends—Dirichlet Boundary Problem) We obtained the solution to

$$(*)\ u_{tt} - c^2 u_{xx} = 0,\ \ u(x,0) = \phi(x)\ \&\ u_t(x,0) = \psi(x) \text{ for } x \in [0,l],\ \ u(0,t) = 0 = u(l,t) \text{ for } t \geq 0,$$

by assuming solutions of the form $u(x,t) = X(x)T(t)$ and then adding these up (infinitely many). The possibility remains that another solution exists which could not be constructed via these separated solutions. Consider the energy for the string at time t:

$$E(t) = \int_0^l c^2 u_x^2 + u_t^2 \, dx.$$

Show that for the solution of $(*)$, this energy is conserved (i.e. $E(t)$ is independent of t) and use this fact to prove the solution of $(*)$ is unique.

3) a) Solve the following problem explicitly in series form:

$$u_{tt} - c^2 u_{xx} = 0,\ \ u(x,0) = x,\ \&\ u_t(x,0) = 0 \text{ for } x \in (0,l),\ \ u_x(0,t) = 0 = u_x(l,t) \text{ for } t \geq 0.$$

b) Fill in the two blanks: The conclusion is that Dirichlet boundary conditions "lead" to an $*******$ extension of the initial data whereas Neumann boundary conditions "lead" to an $*******$ extension of the initial data.

4) (Laplace equation should NOT be interpreted as a time evolution equation). Suppose we consider Laplaces equation is two dimensions and "think of" y as time. Thus we consider the initial value problem:

$$u_{xx} + u_{yy} = 0, \qquad u(x,0) = 0,\ \ u_y(x,0) = \frac{\sin nx}{n},$$

where n is a fixed integer. You can check that this problem has a solution, for each n it is given by

$$u(x,y) = \frac{\sinh ny \ \sin nx}{n^2}.$$

As $n \to \infty$ the initial conditions tend to zero but what happens to the solution? Is this problem well posed?

5) Find all complex eigenvalues of the first derivative operator $\frac{d}{dx}$ subject to the single boundary condition $X(0) = X(1)$. Are the eigenfunctions orthogonal on the interval $(0,1)$?

6)Using Parseval's identity find

$$\sum_{n=1}^{\infty} \frac{1}{n^4}.$$

7) Let $\triangle u = 0$ for x, y in the plane (i.e u is harmonic everywhere). Suppose u is bounded above i.e. there exists a constant M such that $u(x, y) \leq M$ for all x and y. Prove that u must be constant. Interpret this when u represents the temperature at a point (x, y) in the plane. This fact is know as the Liouville theorem.