Math 462, Solutions of test-like problems

During the test the most important thing is to show that you know how to approach the problem. Arithmetical errors is better to avoid, but they are of secondary importance.

We use abbreviation Ex. 1.2.3 means exercise 3 from section 2 of chapter 1 of Weinberger’s book.

1. (Ex. 1.2.9) Derive the solution of the initial boundary-value problem:

\[
\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad \text{for} \quad 0 < x < 1 \quad t > 0
\]

\[
u(x, 0) = f(x) \quad \text{for} \quad 0 \leq x \leq 1
\]

\[
\frac{\partial u}{\partial t}(x, 0) = g(x) \quad \text{for} \quad 0 \leq x \leq 1
\]

\[
u(1, t) = 0 \quad \frac{\partial u}{\partial x}(0, t) = 0.
\]

This problem arises when no vertical force is applied at \( x = 0 \).

Solution: see http://www-users.math.umd.edu/~ssa/462/hw2sol.pdf

2. (Ex. 1.5.5) If

\[
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 1 - x \quad \text{for} \quad 0 < x < 1 \quad t > 0
\]

\[
u(x, 0) = x^2(1 - x) \quad \text{for} \quad 0 \leq x \leq 1
\]

\[
\frac{\partial u}{\partial t}(x, 0) = 0 \quad \text{for} \quad 0 \leq x \leq 1
\]

\[
\frac{\partial u}{\partial x}(0, t) = 0 \quad \frac{\partial u}{\partial x}(0, t) = 0.
\]

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**Solution:** Using the formula for D’Alembert formula we have $g = 0$, $f(x) = x^2(1 - x)$, $F(x, t) = 1 - x$ for $0 \leq x \leq 1$ and

$$u(0.25, 2) = \frac{1}{2} [f(2 + 0.25) + f(0.25 - 2)] +$$

$$\frac{1}{2} \int_0^2 \int_{-3/4 - \bar{t}}^{5/4 - \bar{t}} F(\bar{x}, \bar{t}) \, d\bar{x} \, d\bar{t}.$$  

Thus, we need to determine the extended functions $f(x)$ and $F(\bar{x}, \bar{t})$. Differentiating $u(x, t)$ with respect to $x$ at $(0, t)$ we have

$$\frac{\partial u}{\partial x}(0, t) = \frac{1}{2} [f'(t) - f'(-t)] + \frac{1}{2} \int_0^t [F(\bar{t}, 0) - F(-\bar{t}, 0)] \, d\bar{t} = 0.$$  

Thus, we need to reflect both $f$ and $F$ as even functions:

$$f(x) = f(-x), \quad F(x, t) = F(-x, t) \quad 0 \leq x \leq 1.$$  

Thus, $f(x) = x^2(1 + x)$ and $F(x, t) = 1 + x$ for $-1 \leq x \leq 0$. Check the other boundary condition at $(1, t)$. We have

$$u(1, t) = \frac{1}{2} [f(1 + t) + f(1 - t)] + \frac{1}{2} \int_0^t \int_{1-t}^{1+t} F(\bar{x}, \bar{t}) \, d\bar{x} \, d\bar{t} = 0$$  

Thus, $f$ and $F$ have odd reflection at one:

$$f(x) = -f(2 - x) \quad F(x, t) = -F(2 - x, t), \quad 1 \leq x \leq 2.$$  

This implies

$$f(y) = -(2 - y)^2(1 - (2 - y)) = -(2 - y)^2(1 - y) \quad \text{for} \quad 1 \leq y \leq 2.$$  

Combining the data we have $f$ is given by

$$\begin{align*}
-\frac{(2 + x)^2(1 + x)}{2} & \quad \text{for} \quad -2 \leq x \leq -1 \\
x^2(1 + x) & \quad \text{for} \quad -1 \leq x \leq 0 \\
x^2(1 - x) & \quad \text{for} \quad 0 \leq x \leq 1 \\
-\frac{(2 - x)^2(1 - x)}{2} & \quad \text{for} \quad 1 \leq x \leq 2.
\end{align*}$$

(1)
Away from this integral we can extend $f$ as a periodic function of period 4. Indeed, for $1 \leq x \leq 2$ we have

$$f(x + 2) = -f(2 - (x + 2)) = -f(-x) = f(-x).$$

Notice that when we reflect at $x = 0$ we do not change the sign. Now

$$f(x + 4) = f((x + 2) + 2) = -f(x + 2) = f(x).$$

It is left to plug in $-1.75$ and $2.25$. Due to periodicity they are the same and $f(-1.75) = -(0.25)^2(1 - 1.75) = 1/16 * 3/4 = 3/64$. Analyze now the integral

$$\int_{-1.75}^{2.25} F(\bar{x}, \bar{t}) d\bar{x} d\bar{t}.$$ 

We split it into time intervals such that intersection of $[x - c\bar{t}, x + c\bar{t}]$ with the number integer intervals $[-1, 0]$, $[0, 1]$, etc is constant. To determine this splitting it is useful to draw the $xt$-plane, draw characteristics from the initial condition $(x, t)$ (in our case $(0.25, 2)$) and record intersections with integer vertical lines (in our case $x = -1, 0, 1, 2$). We have

$$\int_0^{0.25} \left[ \int_{-1.75+\bar{t}}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{2.25-\bar{t}} \right] F(\bar{x}, \bar{t}) d\bar{x} d\bar{t}$$

$$+ \int_{0.25}^{0.75} \left[ \int_{-1.75+\bar{t}}^{-1} + \int_{-1}^{0} + \int_{0}^{1} + \int_{1}^{2} + \int_{2}^{2.25-\bar{t}} \right] F(\bar{x}, \bar{t}) d\bar{x} d\bar{t}$$

$$+ \int_{0.75}^{1.25} \left[ \int_{-1.75+\bar{t}}^{0} + \int_{0}^{1} + \int_{1}^{2.25-\bar{t}} \right] F(\bar{x}, \bar{t}) d\bar{x} d\bar{t}$$

$$+ \int_{1.25}^{1.75} \left[ \int_{-1.75+\bar{t}}^{0} + \int_{0}^{2.25-\bar{t}} \right] F(\bar{x}, \bar{t}) d\bar{x} d\bar{t} + \int_{1.75}^{2} \int_{0}^{2.25-\bar{t}} F(\bar{x}, \bar{t}) d\bar{x} d\bar{t}.$$ 

It is useful to look for cancelations, because $F$ has odd reflection at $x = \pm 1$. Renumber all these integrals in the same order they
occur: 1, 2, · · ·, 15. Notice that 3 + 4 = 0, 8 + 5 + 9 = 0 (since $F$ is time independent). We are left with 10 integrals. Due to reflection symmetry integrals 7, 11 are the same. Integral 2 has half of their value. After that tedious calculation give the exact value.

Answer: $u(0.25, 2) = 9/16$.

3. (Ex.1.7.2) Find the domain of dependence of the point $(0.25, 3)$ with respect to the problem

$$
\frac{\partial^2 u}{\partial t^2} - \cot^2 x \frac{\partial^2 u}{\partial x^2} = F(x, t) \quad \text{for} \quad 0 < x < \pi/4 \quad t > 0
$$

$$u(x, 0) = \frac{\partial u}{\partial t}(x, 0) = 0 \quad \text{for} \quad 0 \leq x \leq \pi/4$$

$$u(0, t) = f_1(t) \quad u(\pi/4, t) = f_2(t) \quad \text{for} \quad t > 0.$$

Solution: The characteristics are given by

$$C_1 : t = 2 - \int_{x}^{\pi/8} \frac{d\xi}{\cot \xi}$$

$$C_2 : t = 2 - \int_{\pi/8}^{x} \frac{d\xi}{\cot \xi}$$

Notice that

$$\int_{a}^{b} \frac{d\xi}{\cot \xi} = \int_{a}^{b} \frac{\sin \xi}{\cos \xi} \, d\xi = - \int_{a}^{b} \frac{d \cos \xi}{\cos \xi} = \log \cos a - \log \cos b.$$

Substituting it we get

Answer The domain of dependence is the part of the strip $0 \leq x \leq \pi/4$, where $0 \leq t \leq 2 - |\log \cos x - \log \cos \pi/8|$. 

4. Find the characteristics of

a) $\frac{\partial^2 u}{\partial t^2} - t \frac{\partial^2 u}{\partial x^2}$;

b) $\frac{\partial^2 u}{\partial t^2} + 2e^x \frac{\partial^2 u}{\partial x \partial t} + e^{2x} \frac{\partial^2 u}{\partial x^2} + \cos x \frac{\partial u}{\partial t} + \sin x \frac{\partial u}{\partial x} + x^2 u;$
\[ (\cos^2 x - \sin^2 x) \frac{\partial^2 u}{\partial t^2} + 2 \cos x \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial x^2} + u; \]

Solution

a) Compute the discriminant \( B^2 - 4AC \), \( A = 1 \), \( B = 0 \), \( C = -t \).
We have it equal \( t \). The characteristics \( \xi \) and \( \eta \) satisfy
\[
\frac{dx}{dt} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \pm \sqrt{t}.
\]
Integrating we have
\[
x^{\pm}(t) = x(0) \pm \int_{0}^{t} \sqrt{t} \, dt = x(0) \pm \frac{2}{3} t^{3/2}.
\]
Answer: \( x + \frac{2}{3} t^{3/2} = 2/3 \) and \( x - \frac{2}{3} t^{3/2} = -2/3 \)

b) Compute the discriminant \( B^2 - 4AC \), \( A = 1 \), \( B = 2e^x \), \( C = e^{2x} \).
We have it equal 0. The characteristics \( \xi \) and \( \eta \) coincide and satisfy
\[
\frac{dx}{dt} = \frac{B}{2A} = e^x \quad \text{or} \quad \frac{dt}{dx} = \frac{2A}{B} = e^{-x}.
\]
Integrating we have
\[
t(x) = t(0) + \int_{0}^{x} e^{-x} \, dx = t(0) - e^{-x}.
\]
Answer: \( t = 2 - e^{-x} \).

c) Compute the discriminant \( B^2 - 4AC \), \( A = \cos^2 x - \sin^2 x \), \( B = 2 \cos x \), \( C = 1 \). We have it equal \( 4 \sin^2 x \). The characteristics \( \xi \) and \( \eta \) satisfy
\[
\frac{dt}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2C} = \cos x \pm \sin x.
\]
Integrating we have
\[
t^{\pm}(t) = t(0) \pm \int_{0}^{x} \left( \cos x \pm \sin x \right) \, dx = x(0) \pm (\sin x - \cos x).
\]
Answer: \( t = \sin x + \cos x \) and \( t = 2 + \sin x - \cos x \).
5. Show that the three-dimensional problem

\[ \nabla^2 u = \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = -F(x, y, z) \quad \text{in } D \]

\[ u = f \quad \text{on } C \]

*Solution* Repeat word by word the arguments of Section 11 replacing 2-dimensional integrals there with 3-dimensional ones and 2-dimensional divergence theorem with 3-dimensional one.