MATH 431: PROBLEM SET 4

Let $S \subset \mathbb{R}^3$ be the surface defined by:

$$(1) \quad x^2 + y^2 + z^2 - xyz = 20.$$ 

Let $S_\infty$ be its ideal locus in $\mathbb{P}^3$ and $\tilde{S} := S \cup S_\infty$

1. Write (1) as a homogeneous cubic polynomial in $X, Y, Z, W$.
2. Show that the ideal locus $S_\infty$ is the union of three intersecting (projective) lines in $\mathbb{P}^2$. (Hint: what does $S$ look like at infinity?)
3. Find a set of affine patches to cover $\tilde{S}$. What is the smallest number needed?
4. Show that the surface $\tilde{S} := S \cup S_\infty \subset \mathbb{P}^3$ is smooth, by representing the part of $\tilde{S}$ in an affine patch as a regular level set.
5. Show that the equations
   - $z = 2\sqrt{5}$, $x = (\sqrt{5} + 2)y$;
   - $z = 2\sqrt{5}$, $x = (\sqrt{5} - 2)y$;
   - $z = 2$, $x - y = 4$;
   - $z = 2$, $x - y = -4$;
   - $z = -2$, $x + y = 4$;
   - $z = -2$, $x + y = -4$;
   - $z = -2\sqrt{5}$, $x = - (\sqrt{5} + 2)y$;
   - $z = -2\sqrt{5}$, $x = - (\sqrt{5} - 2)y$

define 8 distinct lines on $S$.
6. Find 16 more lines on $S$.
7. Find 27 lines on $\tilde{S}$.
8. Draw a picture of $S$ illustrating its 24 lines.
9. Projectively transform $\tilde{S}$ to illustrate all 27 lines.
10. (Bonus Problem) An Eckard point is a point where three lines intersect. Find 10 Eckard points on $\tilde{S}$.
11. (Bonus Problem) A tritangent plane is a plane which intersects $\tilde{S}$ in three lines. Find 45 tritangent planes. Must a tritangent plane be tangent to $\tilde{S}$?
12. (Bonus Problem) How many points are intersections of exactly two lines?

Date: 21 November 2006.