Given a point $p$, denote the reflection in the point $p$ by $R_p$. Consider the following three points.

\[
O := \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
p_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}, \\
p_2 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}
\]

(1) Compute the five points:

\[R_O(O), R_O(p_1), R_{p_1}(O), R_{p_2}(p_1), R_{p_2}(p_2).\]

Using the last calculation, verify that $R_O$ and $R_{p_1}$ do not commute:

\[R_O(R_{p_1}(p)) \neq R_{p_1}(R_O(p))\]

for some point $p$.

(2) Let $S_O$ be the rotation through $\pi/2$ about $O$ and $S_{p_1}$ be the rotation through $\pi/2$ about $p_1$. Their composition $S_{p_1} \circ S_O$ is a rotation through some angle $\theta$ about some point $p$. Compute $\theta$ and $p$.

(3) Compute the line $\ell$ passing through $p_1$ and $p_2$.

(4) Compute the reflection $R_\ell$ in $\ell$.

(5) What kind of transformation is the composition $R_\ell \circ S_O$?

(6) Let $\ell$ be the line in $\mathbb{C}$ defined by $\text{Re}(fz) = c$, where $f \in \mathbb{C} \setminus \{0\}$ is a nonzero complex number and $c \in \mathbb{R}$ is a real number. Compute the reflection in $\ell$.

(7) A glide-reflection is an isometry of $\mathbb{C}$ which preserves a geodesic, $\ell$, but reverses orientation. Write a glide-reflection as a complex transformation in terms of $\ell$ and the distance $d$ (its displacement) along which it translates $\ell$.

(8) Show that the composition $R_\ell \circ R_O$ is a glide-reflection and determines its invariant line and its displacement.

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