NONSTANDARD LORENTZ SPACE FORMS

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In their recent paper [8], Kulharni and Raymond show that a closed 3-manifold which admits a complete Lorentz metric of constant curvature 1 (henceforth called a complete Lorentz structure) must be Seifert fibered over a hyperbolic base. Furthermore on every such Seifert fibered 3-manifold with nonzero Euler class they construct such a Lorentz metric. Moreover the Lorentz structure they construct has a rather strong additional property, which they call “standard”: A Lorentz structure is standard if its causal double cover possesses a timelike Killing vector field. Equivalently, it possesses a Riemannian metric locally isometric to a left-invariant metric on SL(2, R). Kulkarni and Raymond asked if every closed 3-dimensional Lorentz structure is standard. This paper provides a negative answer to this question (Theorem 1) and a positive answer to the implicit question raised in [8, 7.1.1] (Theorem 3).

Theorem 1. Let \( M^3 \) be a closed 3-manifold which admits a homogeneous Lorentz structure and satisfies \( H^1(M; \mathbb{R}) \neq 0 \). Then there exists a nonstandard complete Lorentz structure on \( M \).

In [8] it is shown that the unit tangent bundle of a closed surface admits a homogeneous Lorentz structure. Therefore we obtain:

Corollary 2. There exists a complete Lorentz structure on the unit tangent bundle of any closed surface \( F \) of genus greater than one which is not standard.

The homogeneous Lorentz structures are all classified in [8]. A circle bundle of Euler number \( j \) over a closed surface \( F, \chi(F) < 0 \), has a homogeneous structure if and only if \( j|\chi(F) \) (an analogous statement holds when \( M \) has singular fibers, i.e. when \( F \) is an orbifold).

We also show:

Theorem 3. Let \( M^3 \) be a 3-manifold which admits a complete Lorentz structure. Then \( M^3 \) is not covered by a product \( F \times S^1 \), \( F \) a closed surface.

Theorem 3 implies that the Euler class of the Seifert fiber structure of \( M^3 \) is nonzero.
Corollary 4. If a closed 3-manifold $M$ admits a complete Lorentz structure, then $M$ admits a standard Lorentz structure.

In [7] the deformation theory of standard Lorentz structures is extensively discussed.

A key idea in the proof of Theorem 1 is the notion of a (small) deformation of a complete Lorentz structure. It is convenient to think of a Lorentz structure as a “locally homogeneous” geometric structure, defined by an atlas of charts which are homeomorphisms of coordinate patches into a model space $X$ such that the coordinate changes on the overlaps lie in a certain group $G$ of transformations of $X$. (See [12].) In our case $X$ will be a simply connected complete Lorentz manifold of curvature 1 and $G$ will be the identity component of its group of Lorentz isometries. A convenient model for $X$ is the universal cover $\widetilde{\text{SL}}(2, \mathbb{R})$ of $\text{SL}(2, \mathbb{R})$, with the Lorentz metric defined by the Killing form on the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$. The group of all its isometries is a 4-fold extension of the quotient of $\widetilde{\text{SL}}(2, \mathbb{R}) \times \widetilde{\text{SL}}(2, \mathbb{R})$ by a diagonally embedded central $\mathbb{Z}$. See [8] for further details on the resulting geometry.

One basic example of such a structure arises as follows. Consider any discrete cocompact subgroup $\Gamma$ of $\text{PSL}(2, \mathbb{R})$. Then the quotient $\text{PSL}(2, \mathbb{R})/\Gamma$ has an induced left-invariant complete Lorentz structure. Such manifolds have homogeneous Lorentz metrics (cf. Kulkarni-Raymond [8, 10]). If $\Gamma$ is torsionfree, so that $\text{PSO}(2) \backslash \text{PSL}(2, \mathbb{R})/\Gamma$ is a smooth hyperbolic surface $F$, then $\text{PSL}(2, \mathbb{R})/\Gamma$ is the unit tangent bundle of $F$. By taking fiberwise coverings, we obtain homogeneous complete Lorentz structures on other oriented circle bundles over $F$; these circle bundles are characterized by the property that their Euler class divides $\chi(F)$. The class of Seifert fibered 3-manifolds which can be obtained as coverings of such quotients of $\text{PSL}(2, \mathbb{R})$ are precisely the Seifert fibered 3-manifolds which admit homogeneous Lorentz structures. The nonstandard complete Lorentz structures constructed here will be deformations of these homogeneous structures.

A geometric structure modelled on the geometry of $(G, X)$ is sometimes called a “$(G, X)$-structure”. To every $(G, X)$-structure on a manifold $M$ there are associated homomorphisms $h$ from the fundamental group $\pi = \pi_1(M)$ to $G$ such that for each “holonomy homomorphism” $h$ there exists a local diffeomorphism (called the “developing map”) from the universal covering $\tilde{M}$ of $M$ to $X$ which is equivariant respecting $h$. (For a given $(G, X)$-structure, the holonomy homomorphism and the developing map are respectively unique up to conjugation and composition with a transformation in $G$.) If $G$ is a group of isometries of a pseudo-Riemannian metric on $X$, then there is a unique pseudo-Riemannian metric on $M$ such that the developing map is a local
isometry of the induced structure on $\tilde{M}$ with $X$. In the language of [8], a
manifold with a complete Lorentz structure is a “Lorentz space form”.

A $(G, X)$-structure is said to be complete if its developing map is a covering
map onto $X$. We will always take $X$ to be a simply connected homogeneous
space of $G$, so that the developing map will represent a complete $(G, X)$-
manifold as a quotient of $X$ by a discrete subgroup of $G$ acting properly and
freely. When $X$ has a complete $G$-invariant pseudo-Riemannian metric, com-
pleteness of a $(G, X)$-structure is equivalent to the usual notion of geodesic
completeness of the corresponding pseudo-Riemannian metric. However, un-
less $G$ acts properly on $X$ no general criterion for a $(G, X)$-structure on a
closed manifold to be complete is known. (Indeed there are many well-known
geometries $(G, X)$ (such as affine geometry) for which incomplete $(G, X)$-
structures exist on closed manifolds, see e.g. [11].) It is not known whether a
Lorentz structure on a closed manifold is necessarily complete.

A Lorentz structure is standard if it (or perhaps a double cover of it)
possesses a timelike Killing vector field $\xi$. In terms of $(G, X)$-structures a
standard complete Lorentz structure is a $(G, \Lambda_\Gamma)$-structure whose “holonomy
group” $h(\pi)$ normalizes the isometric flow generated by $\xi$. Alternatively we say
that a standard Lorentz structure is a $(G_0, X)$-structure, where $G_0$ is the
normalizer of $\xi$. Every homogeneous Lorentz structure on a closed manifold is
complete (since $G_0$ acts properly on $X$, standard implies complete for closed
manifolds).

The space of homomorphisms $\pi \to G$ forms a real analytic variety
$\text{Hom}(\pi, G)$. Suppose $M$ is a closed manifold with a $(G, X)$-structure (denoted
$M_0$) with holonomy homomorphism $h_0: \pi \to G$. Then there exists a neighbor-
hood $U$ of $h_0$ in $\text{Hom}(\pi, G)$ such that for each $h_t \in U$, there is a $(G, X)$-
structure $M_t$ with holonomy $h_t$. (In this generality, this fact was first observed
by Thurston [12]; See Lok [9] for a detailed discussion.) (Indeed, it is possible
to define a deformation space of $(G, X)$-structures with a natural topology in
such a way that the $(G, X)$-structures $M_t$ form a continuous family near $M_0$.)

Let $M$ be a 3-manifold which admits a homogeneous Lorentz structure, e.g.
the unit tangent bundle of a closed surface $F$. Let $h_0$ be the holonomy
representation $\pi \to \widetilde{\text{SL}}(2, \mathbb{R})$ corresponding to one of the homogeneous Lorentz
structures above. Let $B$ be a one-parameter subgroup in $\text{SL}(2, \mathbb{R})$ acting by
right-multiplication on $\widetilde{\text{SL}}(2, \mathbb{R})$. We shall deform the homomorphism $h_0 \in
\text{Hom}(\pi, G)$ using a deformation of the trivial representation in $\text{Hom}(\pi, B)$.

For $v \in \text{Hom}(\pi, B)$ in a sufficiently small neighborhood of the trivial
homomorphism, the homomorphism $(h_0, v): \pi \to \text{Hom}(\pi, G)$ (where $h_0$ acts
on the left and $v$ acts on the right) will be the holonomy representation of a
complete Lorentz structure near the homogeneous structure on $M$. In other words:

**Proposition 5.** Let $h_0: \pi \to \widetilde{SL}(2, \mathbb{R})$ be the holonomy of a homogeneous complete Lorentz structure as above. Then there exists a neighborhood $U$ of the trivial representation $0$ in $\text{Hom}(\pi, B)$ such that for all $v \in U$, $(h_0, v)$ is a free proper action of $\pi$ on $X$ with quotient a closed manifold.

When $B$ is either a hyperbolic or parabolic one-parameter subgroup, then the resulting quotient manifold has a nonstandard complete Lorentz structure. Thus Proposition 5 implies Theorem 1. Observe that we obtain two quite different families of nonstandard complete Lorentz structures, depending on whether $B$ is parabolic (lightlike) or hyperbolic (spacelike). By small deformations of the holonomy, we construct "nearby" Lorentz structures with the deformed holonomy. Proposition 5 is proved by showing this deformed structure is complete.

We begin by describing one viewpoint on $(G, X)$-structures in which the existence of deformed $(G, X)$-structures is quite transparent. Let $\text{dev}: \tilde{M} \to X$ be a developing map which is equivariant with respect to a homomorphism $h \in \text{Hom}(\pi, G)$. The equivariance of $\text{dev}$ with respect to $h$ implies that the graph of $\text{dev}$ is a section of the trivial $X$-bundle $\tilde{M} \times X$ over $\tilde{M}$ which is invariant under the action of $\pi$ on $\tilde{M} \times X$ defined by $\gamma: (u, x) \mapsto (\gamma u, h(\gamma)x)$. It follows that the graph of $\text{dev}$ defines a section (the "developing section") $f$ of the $(G, X)$-bundle $X(h)$ whose total space is the quotient $(\tilde{M} \times X)/\pi$.

The bundle $X(h)$ has a flat structure, i.e. a foliation transverse to the fibers. The leaves of this foliation are the images of the sets $\tilde{M} \times \{x_0\}$, where $x_0 \in X$. The nonsingularity of the developing map is equivalent to the transversality of $f$ to the flat structure. Conversely, any section of a flat $(G, X)$-bundle which is transverse to the flat structure defines a $(G, X)$-structure: local charts for this structure are found by composing the submersive local charts of the foliation with the section. In this way every transverse section of the flat $(G, X)$-bundle $X(h)$ is a "developing section" of a $(G, X)$-structure with holonomy $h$. For more details on this picture of $(G, X)$-structures, the reader is referred to Goldman [3], Goldman-Hirsch [5], Kulkarni [6], or Sullivan-Thurston [11].

We can now understand the deformation theorem as follows. Fix a $(G, X)$-structure on $M$ as well as a holonomy homomorphism $h_0$, developing section $f_0$ of $X(h_0)$, etc. We will prove that there is a neighborhood $W$ of $h_0$ in $\text{Hom}(\pi, G)$ such that every $h \in W$ is the holonomy of a "nearby" $(G, X)$-structure. First choose a contractible neighborhood $W'$ of $h_0$ in $\text{Hom}(\pi, G)$. Then there is a natural $(G, X)$-bundle over $\tilde{M} \times W'$ whose total space is the
quotient of $\tilde{M} \times W' \times X$ by the action of $\pi$ given by $\gamma: (u, h, x) \mapsto (\gamma u, h, h(\gamma)x)$. The covering homotopy property implies that this bundle is equivalent to the product $X(h_0) \times W'$, as an $X$-bundle. Fix a smooth trivialization of this bundle over $W'$. The foliation defining the flat structure on $X(h)$ varies continuously with respect to $h$ in the $C^1$ topology. Using the trivialization over $W'$, we find a smooth section $f'$ of this bundle over $M$ extending $f_0$. Since $f_0$ is transverse to the flat structure it follows that the restriction $f_i$ of $f'$ to $M \times \{h_i\}$ is also transverse, at least for $h_i$ in a neighborhood $W$ of $h_0$ in $W$. Thus for each $h_i \in W$ there is a $(G, X)$-structure with holonomy $h_i$. We shall refer to the new structures with holonomy $h_i$ as structures “nearby” the original structure with holonomy $h_0$.

We shall need an elementary property of this construction:

**Lemma 6.** Suppose that $M_0$ is a closed $(G, X)$-manifold whose holonomy homomorphism $h$ centralizes a connected subgroup $H$ of $G$ which acts properly and freely on $X$. Consider deformations $M_i$ of $M_0$ induced as above by deformations $h$ of $h_0$ which have the form $h_0(\gamma) = h(\gamma)p_0(\gamma)$, where $p_0$ is a deformation of the trivial representation in $\text{Hom}(\pi, H)$. Let $\text{dev}$ denote the corresponding developing maps of $M_i$, and let $p_H$ denote the projection map $X \to X/H$. Then, as $h$ varies, the composite map $p_H \circ \text{dev}$ remains constant. In particular, if $M_0$ is complete, then $p_H \circ \text{dev}$ is a fibration with fibers the orbits of the corresponding local $H$-action.

**Proof of Lemma 6.** The actions of $\pi$ on the quotient $X/H$ defined by $h_i$ are all equal. The family of associated flat $X/H$-bundles $(X/H)(h_i)$ over $M$ is a bundle over $W$. Since $W$ is contractible, this bundle is trivial. Furthermore there exists a trivialization over $W$ of the family $X(h_i)$ of $X$-bundles over $M$ which extends the trivialization of the associated flat $X/H$-bundles $(X/H)(h_i)$. Let $p_i$ denote the bundle map $X(h_i) \to (X/H)(h_i)$ which on each fiber is given by the projection map $p_H: X \to X/H$. With respect to the trivialization the developing sections, $f_i$ are all equal. Thus $p_i \circ f_i$ is constant in the $t$-parameter. Passing to the universal covering $\tilde{M}$ we see that $p_H \circ \text{dev}$ is constant as well. q.e.d.

**Proof of Proposition 5.** Let $M_0$ be the $(G, X)$-manifold $X/h_0(\pi)$. Let $U$ be a neighborhood of $0$ in $\text{Hom}(\pi, B)$ such that for each $v \in U$, every $(h_0, v)$ is the holonomy of a nearby $(G, X)$-manifold $M_v$. We shall show that $M_v$ is complete.

Let $\text{dev}: \tilde{M} \to X$ denote the developing map of $M_v$. We must show that $\text{dev}$ is bijective. By the lemma, the composition $p_B \circ \text{dev}: \tilde{M} \to X/B$ is equivalent to the composition of $p_B$ with the developing map of $M_0$ and hence is a fibration. Let $\beta_X$ be the Killing vector field on $X$ which generates the action of
B. Let $\beta_M$ be the Killing vector field on $M$ which corresponds to $\beta_X$, i.e. $p^*(\beta_M) = \text{dev}^*(\beta_X)$, where $p : \tilde{M} \to M$ is the covering projection. Since $M$ is compact, the vector field $\beta_M$ is complete and hence $p^*(\beta_M)$ is also complete. Let $\{\phi_s\}_{s \in \mathbb{R}}$ be the flow on $M$ generated by $p^*(\beta_M)$ and $\{\psi_s\}_{s \in \mathbb{R}}$ the flow on $X$ generated by $\beta_X$. Clearly $\text{dev} \circ \phi_s = \psi_s \circ \text{dev}$.

**dev is surjective:** Let $v \in X$. Since $p_B \circ \text{dev}$ is surjective, there exists $u \in \tilde{M}$ such that $p_B(\text{dev}(u)) = p_B(v)$. Since the fibers of $p_B$ are the orbits of $B$ (i.e. the trajectories of $p^*(\beta_M)$), there exists $s \in \mathbb{R}$ such that $\phi_s(\text{dev}(u)) = v$. It follows that $\text{dev}(\psi_s(u)) = v$, as desired.

**dev is injective:** Suppose that $u_0, u_1 \in M$ satisfy $\text{dev}(u_0) = \text{dev}(u_1)$. Since $p_B \circ \text{dev}$ is a fibration with fibers the trajectories of $p^*(\beta_X)$, there exists $s \in \mathbb{R}$ such that $\phi_s(u_0) = u_1$. Thus $\psi_s(\text{dev}(u_0)) = \text{dev}(u_1) = \text{dev}(u_0)$. As $B$ acts freely on $X$, it follows that $s = 0$, and $u_0 = u_1$. Thus dev is bijective and $M$ is complete. q.e.d.

**Remarks.** (i) It seems plausible to conjecture that for every representation $\nu \in \text{Hom}(\pi, \text{SL}(2, \mathbb{R}))$ sufficiently near a standard representation, the homomorphism $(\Lambda^0, \iota)$ defines a properly discontinuous free action on $X$. It would also be interesting to know explicitly, for given $h_0$, which $\nu \in \text{Hom}(\pi, B)$ define properly discontinuous actions.

(ii) By taking $B$ to be a parabolic one-parameter group, we note that the deformation space for complete Lorentz structures is not Hausdorff. Let $(h_0, \nu)$ be a holonomy homomorphism for a nonstandard complete Lorentz structure as above, where $\nu : \pi \to B$. Let $N$ be a hyperbolic one-parameter subgroup normalizing $B$; then the orbit of $(h_0, \nu)$ under conjugation by $N$ on the second factor contains the original homomorphism $(h_0, 1)$ in its closure. Thus the space of equivalence classes of holonomy representations, and hence the deformation space of complete $(G, X)$-structures, is not Hausdorff.

(iii) In a similar way, when $B$ is parabolic every homomorphism $\nu : \pi \to B$ is realized as the second component of the holonomy of a nonstandard Lorentz structure on $M$. For the deformation arguments above realize an open neighborhood $U$ of 1 in $\text{Hom}(\pi, B)$ and every homomorphism $\pi \to B$ is $N$-conjugate to one in $U$.

**Proof of Theorem 3.** Let $M$ be a closed 3-manifold which is a product $F \times S^1$, where $S$ is a closed surface and $\chi(F) < 0$. By [8] the holonomy representation $h : \pi \to G$ composed with the projection

$$p' : G \to G' = G/\text{center}(G) = \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R})$$

is of the form $(h_1, h_2)$, where either $h_1$ or $h_2$ is a Fuchsian representation. We may assume that $h_1$ is Fuchsian. Suppose the genus of $F$ is $g$ and that
\[ \langle A_1, B_1, \ldots, A_g, B_g \mid [A_1, B_1] \cdots [A_g, B_g] = I \rangle \] is the standard presentation for \( \pi' = \pi_1(F) = \pi/\text{center}(\pi) \). (Compare [8, 7.1.1].) Let \( \mu \) be the element of \( \pi \) corresponding to the fiber; since \( \mu \) is central in \( \pi \),
\[ h_1(\mu) \] centralizes \( h_1(\pi) \) and \( h_2(\mu) \) centralizes \( h_2(\pi) \). Since \( h_1(\pi) \) is Fuchsian, \( h_1(\pi) \) must lie in the center of \( \text{PSL}(2, \mathbb{R}) \), i.e. \( h_1(\mu) = 1 \). If \( h_2(\pi) \) is non-abelian, then its centralizer is trivial and \( h_2(\pi) = 1 \). Otherwise \( h_2(\pi) \) is abelian, in which case some power of \( h_2(\mu) \) (which is a product of commutators in \( h_2(\pi) \)), is the identity element of \( \text{PSL}(2, \mathbb{R}) \). Thus some power of \( h(\mu) \) must lie in the center of \( G \). By passing to a finite covering of \( M \) we may assume that \( h(\mu) = 1 \) and that \( h \) factors through a homomorphism \( \pi' \rightarrow G \). Let \( h'_1 \) and \( h'_2 \) be the two components of the composition \( p' \circ h' : \pi \rightarrow \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}) \).

Now consider lifts \( h(A_i) \) of \( h(A_i) \) (respectively \( h(B_i) \) of \( h(B_i) \)) to the universal covering \( G = \text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R}) \) of \( G \). Let \( S = [h(A_1), h(B_1)] \cdots [h(A_g), h(B_g)] \). Since \( h_1 \) is Fuchsian, the projection of \( s \) on the first factor must be \( z^{2-2g} \). Since \( h \) factors through \( \pi' \), the projection of \( s \) on the second factor of this element is also \( z^{2-2g} \). Thus the Euler class of each representation \( h'_1, h'_2 \) equals \( 2 - 2g \). (Compare the proof of Theorem 7.2 in [8], as well as 7.1.1.) We claim that this implies that the Lorentz volume of \( M \) is zero. For the \( G \)-invariant volume form on \( X \) determines a continuous 3-dimensional cohomology class \( \omega \in H^3(BG^\delta) \) such that if \( f : M \rightarrow BG^\delta \) is the classifying map of the flat \( G \)-structure on the tangent bundle, then \( f^*\omega = \text{vol}(M)[M] \). (Here \( G^\delta \) denotes \( G \) with the discrete topology. See [1], [2], [3] and [4] for more information on such classes.) Now the continuous cohomology of \( G \) can easily be computed from the extensions \( Z \rightarrow \tilde{G} \rightarrow G \) and \( Z \rightarrow G \rightarrow \text{PSL}(2, \mathbb{R}) \times \text{PSL}(2, \mathbb{R}) \), in terms of the continuous cohomology of \( \text{PSL}(2, \mathbb{R}) \) and its universal cover \( \text{SL}(2, \mathbb{R}) \). The continuous cohomology of \( \text{PSL}(2, \mathbb{R}) \) has one generator \( a \) in dimension 2 corresponding to the Euler class, and the continuous cohomology of \( \text{SL}(2, \mathbb{R}) \) has one generator \( b \) in dimension 3 corresponding to its bi-invariant volume form. If \( Z \rightarrow S \rightarrow T \) is an extension of groups there is an exact Gysin sequence

\[ \cdots \rightarrow H^i(T) \rightarrow H^i(S) \rightarrow H^{i+2}(S) \rightarrow H^{i+1}(T) \rightarrow \cdots \]

(all the \( H^i \) denoting continuous cohomology), where the first map \( H^i(T) \rightarrow H^i(S) \) is induced from the homomorphism \( S \rightarrow T \) and the second map \( H^i(S) \rightarrow H^{i+2}(S) \) is given by cup product with the characteristic class in \( H^2(S) \) corresponding to the extension \( Z \rightarrow S \rightarrow T \). In the Gysin sequence for
the extension $\mathbb{Z} \to \widetilde{SL}(2, \mathbb{R}) \to PSL(2, \mathbb{R})$ the generator of $H^3(\widetilde{SL}(2, \mathbb{R}))$ corresponding to the invariant volume form maps to the class in $H^2(PSL(2, \mathbb{R}))$ corresponding to the Euler class [2].

Now let $j: \tilde{G} \to \hat{G}$ be the involution given by $(x, y) \to (y, x)$; on $X$, $j$ is represented by an orientation-reversing Lorentz isometry (thinking of $X$ as $\widetilde{SL}(2, \mathbb{R})$, this isometry is just $x \to x^{-1}$). Let $b_1$, $b_2$ be the generators of the continuous cohomology of $G$ coming from the volume forms on each factor. Because $j$ preserves the image of $Z \to G$ and takes the class $\omega \in H^3(G)$ corresponding to Lorentz volume to $-\omega$, we see that the image of $\omega$ under $H^3(G) \to H^3(\hat{G})$ is $b_1 - b_2$.

Now consider the extension $Z \to G \to PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R})$. One sees that the image of $\omega \in H^3(G)$ under the map $H^3(G) \to H^2(PSL(2, \mathbb{R}) \times PSL(2, \mathbb{R}))$ is the class $a_1 - a_2$. It follows that the volume $\omega(h)$ is given (up to a normalizing constant) by the difference of the Euler classes $e(h_1) - e(h_2)$. Thus if $e(h_1) = e(h_2)$, then $\text{vol}(M) = \omega(h) = 0$.

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References


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