3-dimensional affine space forms and hyperbolic geometry

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When can a group $G$ act on $\mathbb{R}^n$ with quotient $M^n = \mathbb{R}^n / G$ a (Hausdorff) manifold?

- $G$ acts by Euclidean isometries $\implies G$ finite extension of a subgroup of translations $G \cap \mathbb{R}^n \cong \mathbb{Z}^k$ (Bieberbach 1912);

- A Euclidean isometry is an affine transformation

$$\bar{x} \stackrel{\gamma}{\mapsto} A\bar{x} + \vec{b}$$

$A \in \text{GL}(n, \mathbb{R}), \vec{b} \in \mathbb{R}^n,$

where the linear part $\mathbb{L}(\gamma) = A$ is orthogonal. ($A \in \text{O}(n)$)

- Only finitely many topological types in each dimension.
- Only one commensurability class.
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  where the linear part $\mathbb{L}(\gamma) = A$ is *orthogonal*. ($A \in \text{O}(n)$)
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A complete affine manifold $M^n$ is a quotient $\mathbb{R}^n/G$ where $G$ is a discrete group of affine transformations.

For $M$ to be a (Hausdorff) smooth manifold, $G$ must act:
- Discretely: ($G \subset \text{Homeo}(\mathbb{R}^n)$ discrete);
- Freely: (No fixed points);
- Properly: (Go to $\infty$ in $G \implies$ go to $\infty$ in every orbit $Gx$).

More precisely, the map

$$G \times X \to X \times X$$

$$(g, x) \mapsto (gx, x)$$

is a proper map (preimages of compacta are compact).

Unlike Riemannian isometries, discreteness does not imply properness.

Equivalently this structure is a geodesically complete torsionfree affine connection on $M$. 
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Margulis Spacetimes

- Most interesting examples: Margulis (∼ 1980):
  - $G$ is a free group acting isometrically on $\mathbb{E}^{2+1}$
    - $\mathbb{L}(G) \subset O(2,1)$ is isomorphic to $G$.
    - $M^3$ noncompact complete flat Lorentz 3-manifold.
  - Associated to every Margulis spacetime $M^3$ is a noncompact complete hyperbolic surface $\Sigma^2$.
  - Closely related to the geometry of $M^3$ is a deformation of the hyperbolic structure on $\Sigma^2$.
- Conjecture: Every Margulis spacetime is diffeomorphic to a solid handlebody.
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Affine space forms and hyperbolic geometry

Geometric 3-manifolds

- Unlike the 8 geometries of Thurston’s Geometrization, affine structures are not Riemannian.
  - No obvious metrics.
  - Usual tools (distance, angle, metric convexity, completeness, volume) NOT available.

**Conjecture:**

A complete affine 3-manifold $M^3 = \mathbb{R}^3 / \Gamma$ is finitely covered by:

- An iterated fibration by cells and circles; or
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Milnor’s Question (1977)

Can a nonabelian free group act properly, freely and discretely by affine transformations on $\mathbb{R}^n$?

- Equivalently (Tits 1971): “Are there discrete groups other than virtually polycyclic groups which act properly, affinely?”
  - If NO, $M^n$ finitely covered by iterated fibration by cells and circles.
  - Dimension 3: $M^3$ compact $\implies M^3$ finitely covered by $T^2$-bundle over $S^1$ (Fried-G 1983),
  - Geometrizable by Euc, Nil or Sol.
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- Connected Lie group $G$ admits a proper affine action $\iff G$ is amenable (compact-by-solvable).
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An idea for a counterexample...

- Clearly a geometric problem: free groups act properly by isometries on $H^3$ hence by diffeomorphisms of $E^3$.
- These actions are \textit{not} affine.
- Milnor suggests:

  Start with a free discrete subgroup of $O(2,1)$ and add translation components to obtain a group of affine transformations which acts freely.

  "However it seems difficult to decide whether the resulting group action is properly discontinuous."
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Lorentzian and Hyperbolic Geometry

- $\mathbb{R}^{2,1}$ is the 3-dimensional real vector space with inner product:
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  and Minkowski space $\mathbb{E}^{2,1}$ is the corresponding affine space, a simply connected geodesically complete Lorentzian manifold.

- The Lorentz metric tensor is $dx^2 + dy^2 - dz^2$.

- $\text{Isom}(\mathbb{E}^{2,1})$ is the semidirect product of $\mathbb{R}^{2,1}$ (the vector group of translations) with the orthogonal group $O(2,1)$.

- The stabilizer of the origin is the group $O(2,1)$ which preserves the hyperbolic plane
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- Generators $g_1, g_2$ pair half-spaces $A_i^\pm \rightarrow H^2 \setminus A_i^\pm$.
- $g_1, g_2$ freely generate discrete group.
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Early 1980’s: Margulis tried to answer Milnor’s question negatively but instead proved that nonabelian free groups can act properly, affinely on $\mathbb{R}^3$. 
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Flat Lorentz manifolds

Suppose that $\Gamma \subset \text{Aff}(\mathbb{R}^3)$ acts properly and is not solvable.

- Let $\Gamma \xrightarrow{L} \text{GL}(3, \mathbb{R})$ be the linear part.
  - $L(\Gamma)$ (conjugate to) a discrete subgroup of $\text{O}(2, 1)$;
  - $L$ injective. (Fried-G 1983).
- Homotopy equivalence
  \[ M^3 := \mathbb{R}^{2,1}/\Gamma \longrightarrow \Sigma := \mathbb{H}^2/L(\Gamma) \]
  where $\Sigma$ complete hyperbolic surface.
  - Mess (1990): $\Sigma$ not compact.
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Closed geodesics and holonomy

- Each such element leaves invariant a unique (spacelike) line, whose image in $\mathbb{E}^{2,1}/\Gamma$ is a closed geodesic. Just as for hyperbolic surfaces, most loops are freely homotopic to closed geodesics.

$$\gamma = \begin{bmatrix} e^{\ell(\gamma)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\ell(\gamma)} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha(\gamma) \\ 0 \end{bmatrix}$$

- $\ell(\gamma) \in \mathbb{R}^+:$ geodesic length of $\gamma$
- $\alpha(\gamma) \in \mathbb{R}:$ (signed) Lorentzian length of $\gamma$.
- The unique $\gamma$-invariant geodesic $C_\gamma$ inherits a natural orientation and metric and $\gamma$ translates along $C_\gamma$ by $\alpha(\gamma)$. 
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- In affine space, half-spaces disjoint $⇒$ parallel!
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Images of crooked planes under a linear cyclic group.

The resulting tessellation for a linear boost.
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Affine action of level 2 congruence subgroup of $\text{GL}(2, \mathbb{Z})$

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Marked Signed Lorentzian Length Spectrum

- For every affine deformation \( \Gamma \xrightarrow{\rho=(L,u)} \text{Isom}(\mathbb{E}^{2,1})^0 \), define \( \alpha_u(\gamma) \in \mathbb{R} \) as the (signed) displacement of \( \gamma \) along the unique \( \gamma \)-invariant geodesic \( C_\gamma \), when \( L(\gamma) \) is hyperbolic.
- \( \alpha_u \) is a class function on \( \Gamma \);
- When \( \rho \) acts properly, \( |\alpha_u(\gamma)| \) is the Lorentzian length of the closed geodesic in \( M^3 \) corresponding to \( \gamma \);
- (Margulis 1983) If \( \rho \) acts properly, either
  - \( \alpha_u(\gamma) > 0 \ \forall \gamma \neq 1 \), or
  - \( \alpha_u(\gamma) < 0 \ \forall \gamma \neq 1 \).
- The Margulis invariant \( \Gamma \xrightarrow{\alpha} \mathbb{R} \) determines \( \Gamma \) up to conjugacy (Charette-Drumm 2004).
Marked Signed Lorentzian Length Spectrum

- For every affine deformation $\Gamma \xrightarrow{\rho=(L,u)} \text{Isom}(\mathbb{E}^{2,1})^0$, define $\alpha_u(\gamma) \in \mathbb{R}$ as the (signed) displacement of $\gamma$ along the unique $\gamma$-invariant geodesic $C_\gamma$, when $L(\gamma)$ is hyperbolic.
- $\alpha_u$ is a class function on $\Gamma$;
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- (Margulis 1983) If $\rho$ acts properly, either
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Affine deformations

Start with a Fuchsian group $\Gamma_0 \subset \text{O}(2,1)$. An affine deformation is a representation $\rho = \rho_u$ with image $\Gamma = \Gamma_u$ determined by its translational part $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$.

Conjugating $\rho$ by a translation $\iff$ adding a coboundary to $u$.

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Deformations of hyperbolic structures

- Translational conjugacy classes of affine deformations of $\Gamma_0$ \(\leftrightarrow\) infinitesimal deformations of the hyperbolic surface $\Sigma$.

- The Lorentzian vector space $\mathbb{R}^{2,1}$ corresponds to the Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ with the Killing form, and the action of $O(2,1)$ is the adjoint representation.

- This Lie algebra comprises the Killing vector fields, infinitesimal isometries, of $H^2$.

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Deformations of hyperbolic structures
Deformation-theoretic interpretation of Margulis invariant

- Suppose \( u \in Z^1(\Gamma_0, \mathbb{R}^{2,1}) \) defines an \textit{infinitesimal deformation} tangent to a smooth deformation \( \Sigma_t \) of \( \Sigma \).
  - The marked length spectrum \( \ell_t \) of \( \Sigma_t \) varies smoothly with \( t \).
  - Margulis’s invariant \( \alpha_u(\gamma) \) represents the derivative

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\left. \frac{d}{dt} \right|_{t=0} \ell_t(\gamma)
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(G-Margulis 2000).
- \( \Gamma_u \) is proper \( \implies \) all closed geodesics lengthen (or shorten) under the deformation \( \Sigma_t \).
- When \( \Sigma \) is homeomorphic to a three-holed sphere, the converse holds. (Jones 2004, Charette-Drumm-G 2009).
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Affine space forms and hyperbolic geometry

Extensions of the Margulis invariant

- \( \alpha_u \) extends to parabolic \( \mathbb{L}(\gamma) \). (Charette-Drumm 2005).
- (Margulis 1983) \( \alpha_u(\gamma^n) = |n|\alpha_u(\gamma) \).
  - Therefore \( \alpha_u(\gamma)/\ell(\gamma) \) is constant on cyclic (hyperbolic) subgroups of \( \Gamma \).
  - Such cyclic subgroups correspond to periodic orbits of the geodesic flow \( \Phi \) of \( U\Sigma \).
- The Margulis invariant extends to a continuous functional \( \Psi_u(\mu) \) on the space \( C(\Sigma) \) of \( \Phi \)-invariant probability measures \( \mu \) on \( U\Sigma \). (G-Labourie-Margulis 2009)

- When \( \mathbb{L}(\Gamma) \) is convex cocompact, \( \Gamma_u \) acts properly \( \iff \) \( \Psi_u(\mu) \neq 0 \) for all invariant probability measures \( \mu \).
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The Deformation Space

- The deformation space of marked Margulis space-times arising from a topological surface $S$ with finitely generated fundamental group is a bundle over the Fricke space $\mathcal{F}(S)$ of marked hyperbolic structures $S \rightarrow \Sigma$ on $S$.
  - The fiber is the subspace of $H^1(\Sigma, \mathbb{R}^{2,1})$ (equivalence classes of all affine deformations) consisting of proper deformations of the fixed hyperbolic surface $\Sigma$.
  - It is nonempty (Drumm 1990).
  - (G-Labourie-Margulis 2010) Convex domain in $H^1(\Sigma, \mathbb{R}^{2,1})$ defined by the generalized Margulis-invariants of measured geodesic laminations on $\Sigma$.
- Thus the deformation space is a cell with some boundary components corresponding to the ends of $S$. 
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The three-holed sphere (Charette-Drumm-G 2009)

- Suppose $\Sigma$ is a three-holed sphere with boundary $\partial_1, \partial_2, \partial_3$.
- Charette-Drumm-Margulis-invariants of $\partial \Sigma$ identify the deformation space $H^1(\Gamma_0, \mathbb{R}^{2,1})$ of equivalence classes of all affine deformations with $\mathbb{R}^3$.
- If $\alpha(\partial_i) > 0$ for $i = 1, 2, 3$, then $\Gamma$ admits a crooked fundamental polyhedron:
  - $\Gamma$ acts properly;
  - $M^3$ is a solid handlebody of genus two.
- Corollary (in hyperbolic geometry): If each component of $\partial \Sigma$ lengthens, then every curve lengthens under a deformation of the hyperbolic surface $\Sigma$. 
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If $\alpha(\partial_i) > 0$ for $i = 1, 2, 3$, then $\Gamma$ admits a crooked fundamental polyhedron:
- $\Gamma$ acts properly;
- $M^3$ is a solid handlebody of genus two.

Corollary (in hyperbolic geometry): If each component of $\partial \Sigma$ lengthens, then every curve lengthens under a deformation of the hyperbolic surface $\Sigma$. 
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Linear functionals $\alpha(\gamma)$ when $\Sigma$ is a three-holed sphere.

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Properness region bounded by infinitely many intervals, each corresponding to a simple loop on $\Sigma$. Boundary points lie on intervals or are points of strict convexity (irrational laminations) (G-Margulis-Minsky).
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Questions for the future

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- Is every nonsolvable complete affine 3-manifold $M^3$ a solid handlebody?
- Which $\mu \in C(\Sigma)$ maximize (minimize) the generalized Margulis invariant?
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