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CHARACTERISTIC CLASSES AND REPRESENTATIONS OF DISCRETE SUBGROUPS OF LIE GROUPS

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A volume invariant is used to characterize those representations of a countable group into a connected semisimple Lie group G which are injective and whose image is a discrete cocompact subgroup of G. Let π be a discrete cocompact subgroup of G and consider the analytic variety $\mathrm{Hom}(\pi, G)$ consisting of homomorphisms $\phi\colon \pi \longrightarrow G$. Denote by K a maximal compact subgroup of G and $X = K\backslash G$ the associated symmetric space. Let M be the orbit space X/π.

(For convenience we shall henceforth assume that π is *torsionfree*: by Selberg's lemma [12] this may be accomplished by replacing π by a subgroup of finite index. This insures that M is a compact smooth manifold having π as its fundamental group. The case when π has torsion follows from the torsionfree case with minor modifications but these modifications need not concern us here.)

To every representation $\phi \in \mathrm{Hom}(\pi, G)$ we associate a foliated bundle E_ϕ over M with fibre X and structure group G (see e.g. [6]). If ω is a closed G-invariant differential k-form on X then we may spread ω over the fibres of E_ϕ (copies of X) to obtain a closed k-form ω_ϕ on E_ϕ. We define $\omega(\phi) = \int_M f^*\omega_\phi$ where f is any section[1] of E_ϕ. Moreover $\omega(\phi)$ is independent of the choice of section. For example taking ω to be the G-invariant volume form on X we obtain a real number $v(\phi)$ which depends on ϕ.

When X is even-dimensional the Chern-Gauss-Bonnet theorem implies that $v(\phi)$ may be described as an Euler number, i.e. the self-intersection number of any section, which is a topological invariant of E_ϕ. When X is odd-dimensional this volume invariant is related to a more recent kind of topological invariant (based on bounded cohomology and due to Gromov [3]) and is constant on the connected[2] components of $\mathrm{Hom}(\pi, G)$.

The volume invariant satisfies an inequality

$$(*) \qquad |v(\phi)| \leqslant \mathrm{volume}(M)$$

(where $\mathrm{volume}(M) = |v(i)|$, $i\colon \pi \longrightarrow G$ is the identity). For $G = PSL(2; \mathbf{R})$ we recover the famous inequality of Milnor [9] and Wood [17] bounding the Euler number of circle bundles over surfaces admitting flat structures.

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[1] Sections exist and are all homotopic since X is contractible.

[2] In the "usual" topology, not the Zariski topology.

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CONJECTURE A. Equality holds in (*) if and only if ϕ is an isomorphism of π onto a discfete subgroup of G.

We state this as a conjecture since we know it presently except for certain G of rank 1. For G compact it is obvious. When $\mathbf{R}$-rank$((G) > 1$ and $\pi \subset G$ is an *irreducible*[3] lattice the conjecture may be deduced from Margulis' "superrigidity" theorem [8] (see also [18]) as follows. Margulis proves, under the assumptions on π and G above, that unless a homomorphism $\phi: \pi \longrightarrow G$ is an isomorphism onto a discrete subgroup (in which case it differs from i by an inner automorphism) the image $\phi(\pi)$ is precompact. In that case there exists $h \in G$ so that $\phi(\pi) \subset h^{-1}Kh$ whence $\phi(\pi)$ fixes a point $x \in X$. Letting f be the section of E_ϕ which is the leaf corresponding to x we obtain $f^*\omega_\phi = 0$ whence $v(\phi) = 0$.

For G locally isomorphic to $SO(n, 1)$ (X = hyperbolic n-space), Conjecture A may be proved along the lines of Thurston's generalization of Gromov's proof of Mostow rigidity [14, §6.4, pp. 6.15–6.18] (combined with [5] for the case $n > 3$). See also Gromov's Bourbaki seminar [4].

Now we specialize to the case G is locally isomorphic to $PSL(2, \mathbf{R})$. For a detailed elementary proof of Conjecture A in this case see [2]. It is interesting to note that the number of values assumed by $v(\phi)$, $\phi \in \mathrm{Hom}(\pi, G)$ is unbounded over all choices π—sharply contrasting the corollary of Margulis' theorem above.

We will state a formula for the number of connected components of $\mathrm{Hom}(\pi, G)$ in terms of the *genus* g of the compact Riemann surface M. It is a general observation that for Γ finitely generated and G an algebraic Lie group the space $\mathrm{Hom}(\Gamma, G)$ is an algebraic variety and has finitely many connected components[4] – a fact already used in characteristic class discussions (Lusztig, see Sullivan [13] and Gromov [3]).

THEOREM B. *The map* v: $\mathrm{Hom}(\pi, PSL(2, \mathbf{R})) \longrightarrow \mathbf{R}$ *induces an isomorphism of the set of connected components of* $\mathrm{Hom}(\pi, PSL(2, \mathbf{R}))$ *onto* $\{2\pi n$: $n \in \mathbf{Z}$ *and* $|n| \leqslant 2g - 2\}$.

In particular there are $4g - 3$ components. On the other hand there are only two *irreducible* components, in the sense of a real algebraic variety. Two of the connected components, corresponding to the maximum and minimum volume and related by changing the orientation of M, consist entirely of faithful discrete representations.[5] Each such component is the space investigated by Weil [15], which is a principal G-bundle over the Teichmüller space of M.

[3] When $\pi \subset G$ is not irreducible the conjecture follows once it is known for the irreducible factors of π.

[4] This is also true if G is semisimple with finite center.

[5] In general the subset of $\mathrm{Hom}(\pi, G)$ consisting of faithful discrete representations is closed (by [7], see also [11, 5.10]) and if $\pi \subset G$ is cocompact also open (by [15], see also [14, 5.1].

THEOREM C. *Let G be the n-fold covering group of $PSL(2, \mathbf{R})$ and π the fundamental group of a surface of genus g. Then the number of connected components of* $\mathrm{Hom}(\pi, G)$ *is given by the following formula*:

$$\begin{array}{ll} 2n^{2g} + (4g-4)/n - 1, & \text{if } n \mid 2g-2, \\ 2[(2g-2)/n] + 1, & \text{if } n \nmid 2g-2. \end{array}$$

Due to their special significance we briefly mention a few results concerning the components of $\mathrm{Hom}(\pi, G)$ when π is a surface group but G is not locally isomorphic to $PSL(2, \mathbf{R})$. For example $\mathrm{Hom}(\pi, G)$ has two components for $G = PSL(2, \mathbf{C})$ and $SO(3)$ but $\mathrm{Hom}(\pi, G)$ is connected for $G = SU(2)$ (Newstead **[10]**), $SL(2, \mathbf{C})$, and any 1-connected 3-dimensional Lie group. However if G is not an algebraic Lie group, then $\mathrm{Hom}(\pi, G)$ may have infinitely many components: for the simplest example take G locally isomorphic to the Heisenberg group but not simply connected [2].

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