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Local rigidity of discrete groups acting on complex hyperbolic space

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Introduction

The superrigidity theorem of Margulis, see Zimmer [17], classifies finite dimensional representations of lattices in semi-simple Lie groups of real rank strictly larger than 1. It is a fundamental problem to obtain the classification of finite dimensional representations of lattices in rank 1 semi-simple Lie groups. That this problem will be considerably harder than the previous one is suggested by the existence of continuous families of inequivalent representations or, in other words, the existence of non-trivial deformations. In Johnson-Millson [6] and Kourouniotis [8] such deformations were constructed for certain representations of lattices in $SO(n, 1)$ based on a construction of Thurston called bending. The deformation space of the representation of a lattice $\Gamma \subset SO(n, 1)$ obtained by restricting an inclusion $SO(n, 1) \to SO(n+1, 1)$ to Γ is of particular interest. If $n > 2$ the space of infinitesimal deformations is $H^1(\Gamma, \mathbf{R}^{n+1})$ where Γ acts on $\mathbf{R}^{n+1}$ by the restriction of the standard action of $SO(n, 1)$. The space of infinitesimal deformations is non-zero for the standard arithmetic examples and the main point of the papers cited above was to establish that some of these infinitesimal deformations are integrable (in [6] it is also shown that some are not).

In this paper, we study the complex analogue of the above example. We let Γ be a cocompact torsion free lattice in $SU(n, 1)$ and consider the deformation space of the representation of Γ obtained by restricting an inclusion $SU(n, 1) \to SU(n+1, 1)$. If $n > 1$ the space of infinitesimal deformations is $H^1(\Gamma, \mathbb{R}) \oplus H^1(\Gamma, \mathbf{C}^{n+1})$. In the first summand Γ acts trivially and the infinitesimal deformations are tangent to the obvious deformations obtained by deforming Γ in $U(n, 1)$ by a curve of homomorphisms into the center of $U(n, 1)$ (observe that the above inclusion factors as $SU(n, 1) \to U(n, 1) \to SU(n+1, 1)$). In the second summand Γ acts by the restriction of the standard action of $SU(n, 1)$. This summand is non-zero for the standard arithmetic examples,

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Borel-Wallach [1], 5.9 and is the complex analogue of the space of infinitesimal deformations in the real case considered above. Thus, it is rather surprising that *none* of these latter infinitesimal deformations are integrable (Theorem 4.2 of this paper). This fact is the main technical result behind our main theorem. We now give two statements of our main theorem; the first one geometric, the second one group-theoretic.

Let D^{n+1} be the unit ball in $\mathbf{C}^{n+1}$ equipped with the metric of constant holomorphic sectional curvature-1 (complex hyperbolic space).

Theorem 1. *Let Γ be a torsion free group acting isometrically and properly discontinuously on D^{n+1} in such a way as to stabilize a totally geodesic n-ball D^n. Assume the quotient $\Gamma \backslash D^n$ is compact. Then all nearby isometric actions of Γ on D^{n+1} also stabilize a totally geodesic n-ball.*

We now give the group-theoretic version of our theorem. We define a representation ρ from Γ into $SU(n+1, 1)$, see Sect. 1 for notation, to be *Fuchsian* if it is discrete and faithful and leaves invariant a line in $\mathbf{C}^{n+2}$ containing a vector of positive length for the hermitian form of signature $(n+1, 1)$. Equivalently, ρ is Fuchsian if it leaves invariant a totally geodesic n-ball contained in the $n+1$-ball. We can now state our main theorem.

Theorem 2. *The set of Fuchsian representations is a connected component in the strong (or classical) topology on* $\operatorname{Hom}(\Gamma, SU(n+1, 1))$.

In fact we give a complete description of the component. It is a locally algebraic subset diffeomorphic to the smooth manifold $(SU(n+1, 1)/S^1) \times T$ where T is the torus $\operatorname{Hom}(\Gamma, S^1)$ and S^1 is embedded as the center Z of the subgroup (isomorphic to $U(n, 1)$) of $SU(n+1, 1)$ stabilizing the first standard basis vector. In particular, we observe that if ρ_t is any curve in $\operatorname{Hom}(\Gamma, SU(n+1, 1))$ containing a Fuchsian representation ρ then there exist g_t in $SU(n+1, 1)$ and χ_t in $\operatorname{Hom}(\Gamma, Z)$ such that $\rho_t = \operatorname{Ad} g_t \circ (\rho \chi_t)$.

We now sketch the proof of our theorem. We let $\mathscr{R}_0$ denote the set of Fuchsian representations. In the first section we assume $\mathscr{R}_0$ is not open and use the Curve Selection Lemma of F. Bruhat and H. Cartan, see Milnor [11], to find a curve of representations ρ_t intersecting $\mathscr{R}_0$ at a single point ρ. We also prove that $\mathscr{R}_0$ is closed in $\operatorname{Hom}(\Gamma, SU(n+1, 1))$.

In Sect. 2 we construct a quadratic form Q on $H^1(\Gamma, \mathbf{C}^{n+1})$. Here $\mathbf{C}^{n+1}$ is identified with the orthogonal complement in $\mathbf{C}^{n+2}$ of the invariant line L of ρ and Γ operates by the action induced by ρ twisted by a unitary character. The form Q is defined as follows. The given Hermitian form on $\mathbf{C}^{n+2}$ induces a form of signature $(n, 1)$ on the orthogonal complement of L. Using the imaginary part of this form we construct a symmetric bilinear form $A(\cdot, \cdot)$ on $H^1(\Gamma, \mathbf{C}^{n+1})$ with values in $H^2(\Gamma, \mathbf{R})$. But $H_2(\Gamma, \mathbf{R})$ contains a canonical class Z, the $(n-1)$-fold self-intersection of the hyperplane section class. We define $Q(c)$, for $c \in H^1(\Gamma, \mathbf{C}^{n+1})$, to be the Kronecker index of $A(c, c)$ and Z. In Sect. 2 we prove by examining the Taylor expansion of ρ_t around ρ that the assumption that $\mathscr{R}_0$ is not open implies that Q is isotropic.

In the last two sections we prove that Q is in fact positive definite contradicting the assumption that $\mathscr{R}_0$ was not open. The fact that Q is positive definite turns out to be equivalent to a special case of a general vanishing theorem of Matsushima-Murakami [9]. In the case $n=1$, it was proved in Goldman [5] by a different method that Q was positive definite. Also Theorem 2 was proved in [5] again by a different method for the case $n=1$. For this reason we may make the assumption in this paper that $n \geqq 2$.

As remarked at the end of this paper, our results generalize to the case of a torsion-free group acting isometrically and properly discontinuously on D^{n+k} stabilizing a totally geodesic D^n such that $\Gamma \backslash D^n$ is compact. We again call such an action of Γ *Fuchsian*. It seems reasonable to look for a "strong rigidity" theorem corresponding to our "local rigidity" results. However, it is important to observe that there is an invariant (under conjugation by $SU(n+k, 1)$) of isometric actions of Γ on D^{n+k} obtained as follows. The Kahler form ω on D^{n+k} (properly normalized) corresponds, by the van Est Theorem, to a continuous Eilenberg-MacLane cohomology class $\omega \in H^2(SU(n+k, 1), \mathbf{R})$ which is the image of an integral class. Taking exterior powers we get a class $\omega^n \in H^{2n}(SU(n+k, 1), \mathbf{R})$. If ρ is a representation of Γ in $SU(n+k, 1)$ we can pull back ω^n by ρ to obtain a class $\rho^* \omega^n$ in $H^{2n}(\Gamma, \mathbf{R})$. Evaluating $\rho^* \omega^n$ on the fundamental class in $H_{2n}(\Gamma, \mathbf{Z})$ we obtain an integer which we denote $v(\rho)$ – in the case ρ is Fuchsian $v(\rho)$ is the volume of $\Gamma \backslash D^n$. We observe that v is constant on each component of $\operatorname{Hom}(\Gamma, SU(n+k, 1))$.

We now conjecture that *if* $\rho \in \operatorname{Hom}(\Gamma, SU(n+k, 1))$ *satisfies* $v(\rho) = \operatorname{vol}(\Gamma \backslash D^n)$ *then* ρ *is Fuchsian.* There is considerable evidence in addition to the main result of this paper that this strong rigidity result should hold. The conjecture has been proved for the case $n=1$ and all k in unpublished work of D. Toledo. Also if ρ is a representation of Γ into $SU(n+k, 1)$ there is a Γ-equivariant smooth map $f: D^n \to D^{n+k}$. Results of Faran [3] and Webster [15] show that it suffices to prove that f is a holomorphic embedding.

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1. Fuchsian representations and their deformations

In this section we prove that the set of Fuchsian representations is a locally algebraic subset of $\operatorname{Hom}(\Gamma, G)$. That is, we construct a real algebraic subset X of $\operatorname{Hom}(\Gamma, G)$ and a strongly open subset U of $\operatorname{Hom}(\Gamma, G)$ such that $X \cap U$ is the set of Fuchsian deformations. X will consist of the representations which have an invariant line and U will consist of the representations which do not leave invariant a null-line or a negative line (in fact U is slightly smaller than this). We then assume that the set of Fuchsian representations is not open in $\operatorname{Hom}(\Gamma, G)$. We apply the Curve Selection Lemma to deduce the existence of a real analytic curve ρ_t in U meeting the space of Fuchsian representations at a single point.

We now establish some notation. In what follows G will denote the Lie group $SU(n+1, 1)$ the subgroup of the unimodular matrices consisting of the (unimodular) isometries of the hermitian form $F: \mathbf{C}^{n+2} \to \mathbf{R}$ given by:

$$F(z_1, \ldots, z_{n+2}) = |z_1|^2 + \ldots + |z_{n+1}|^2 - |z_{n+2}|^2.$$

We let $\langle , \rangle$ denote the sesquilinear form associated to F. We will assume throughout that $n \geqq 2$. We will sometimes let W denote $\mathbf{C}^{n+2}$.

We now give a structure of a real algebraic group to the complexification $\underline{G}$ of G so that the associated real points are G. We note $\underline{G} = SL_{n+2}(\mathbf{C})$. It is sufficient to define an antilinear action of $\mathbf{Z}/2$ (the Galois group of $\mathbf{C}$ over $\mathbf{R}$) on $\underline{G}$. We let r be the $n+2$ by $n+2$ diagonal matrix with diagonal entries $(1, \ldots, 1, -1)$. We make the non-trivial element of $\mathbf{Z}/2$ act by the anti-holomorphic involution σ defined by:

$$\sigma(g) = r(g^*)^{-1} r.$$

Here g^* is the usual adjoint of g; that is, $g^* = {}^t\bar{g}$ where the superscript t denotes transpose. We have now given a real structure to $SL_{n+2}(\mathbf{C})$ - a polynomial function f on $SL_{n+2}(\mathbf{C})$ is real if:

$$\overline{f(\sigma(g))} = f(g).$$

Here $\overline{f(\sigma(g))}$ denotes the complex conjugate of the complex number $f(\sigma(g))$. Clearly the set of real points of $SL_{n+2}(\mathbf{C})$ for the above real structure is $SU(n+1, 1)$. We often abuse notation by letting the same symbol denote both the set of complex points and the algebraic group. We may use the involution σ to give a real structure to $\underline{G}^N$ for any integer $N > 1$. Then if Γ is a group generated by N elements we obtain an induced real structure on $\mathrm{Hom}(\Gamma, \underline{G})$.

In what follows $\underline{G}_0$ will denote the real algebraic subgroup of $\underline{G}$ consisting of those elements which leave invariant the line through e_1 where $\{e_1, e_2, \ldots, e_{n+2}\}$ is the standard basis for $\mathbf{C}^{n+2}$. We will denote by V the subvector space of $\mathbf{C}^{n+2}$ consisting of those vectors with first component zero. We let G_0 denote the real points of $\underline{G}_0$. Then G_0 is isomorphic to $U(n, 1)$ and the induced action of G_0 on V is equivalent to the standard representation of $U(n, 1)$ on $\mathbf{C}^{n+1}$. We will often identify G_0 with $U(n, 1)$ and V with $\mathbf{C}^{n+1}$. We will also use $\langle , \rangle$ to denote the induced sesquilinear form on V.

Let Γ be a torsion-free group and $\rho_0 \colon \Gamma \to U(n, 1)$ be a faithful representation of Γ as a cocompact discrete subgroup of $U(n, 1)$. Then Γ operates via ρ_0 on the unit ball D^n in $\mathbf{C}^n$ as isometries of the Bergmann metric, see Kobayashi-Nomizu [7], p. 282. In [7], D^n, together with the Bergmann metric is called *complex hyperbolic space*. We have adopted this terminology in our title. The quotient space $M = \Gamma \backslash D^n$ is a compact Kahler manifold. A suitable multiple of ω, the imaginary part of the Bergmann metric, is the Chern form of a line bundle over M which is a fortiori positive. Hence, by the Kodaira Embedding Theorem, there exists a complex analytic embedding from M into a complex projective space $\mathbf{P}^N(\mathbf{C})$. A suitable multiple of ω is the Poincare dual of the hyperplane section class in M, the homology class determined by the intersection of the image of M with a generic hyperplane in $\mathbf{P}^N(\mathbf{C})$. We now investigate the connected component of $\mathrm{Hom}(\Gamma, U(n, 1))$ containing ρ_0.

Since $\rho_0(\Gamma)$ is torsion free we see that the intersection of $\rho_0(\Gamma)$ and the center of $U(n, 1)$ is $\{e\}$. Consequently $\rho_0(\Gamma)$ projects isomorphically into the simple Lie group $PU(n, 1)$. We now prove a general theorem which will imply that for any Lie group G the set of discrete faithful representations of Γ into G

form a closed subset of $\mathrm{Hom}(\Gamma, G)$. We observe that $\rho_0(\Gamma)$ is Zariski dense in $PU(n, 1)$ by the Borel Density Theorem [12], Chap. V.

Theorem 1.1. *Let Γ be a torsion free subgroup of a connected semi-simple linear Lie group H. Assume Γ is Zariski dense in H. Let G be any linear Lie group. Then the discrete faithful representations of Γ into G form a closed subset of* $\mathrm{Hom}(\Gamma, G)$.

The theorem is a consequence of the next two lemmas.

Lemma 1.1. *Suppose Γ is a torsion free group containing no non-trivial nilpotent normal subgroups. Let G be a linear Lie group. Then the set of discrete faithful representations is closed in* $\mathrm{Hom}(\Gamma, G)$.

Proof. Choose a neighbourhood U of the identity in G such that if Γ is any discrete subgroup of G then $\Gamma \cap U$ is contained in a connected nilpotent Lie subgroup of G, see Raghunathan [12], Theorem 8.16.

Now let $\{\rho_n\}$ be a sequence of discrete faithful representations converging to a representation ρ. We first show that ρ is faithful. Let $K = \mathrm{Ker}\,\rho$. Then K is normal. We claim K is nilpotent. Since ρ_n embeds Γ into a linear Lie group and Γ is torsion-free, there is an upper bound to the length of the descending central series and it is sufficient to show every finitely generated subgroup of K is nilpotent. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ be a finite subset of K. Then there exists n so that $\{\rho_n(\gamma_1), \ldots, \rho_n(\gamma_r)\} \subset U$. Consequently the image of the subgroup K' generated by $\{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ is nilpotent. Hence K' is nilpotent and consequently K is also. Hence $K = \{e\}$ and ρ is faithful.

We now prove ρ is discrete. Suppose $\rho(\Gamma)$ is not discrete. Let N be the closure of $\rho(\Gamma)$ in G so N is a Lie group. Let N^0 be the connected component of the identity in N. Then $N^0 \neq \{e\}$ for otherwise N would be discrete and consequently $\rho(\Gamma)$ would also be discrete. Now N^0 is generated by any neighborhood of the identity hence by $N^0 \cap U$. Consequently $U \cap \Gamma$ generates a dense subgroup of N^0 and N^0 is nilpotent. Since N^0 is normal in N we see $\rho^{-1}(N^0)$ is normal in Γ. But since ρ is faithful $\rho^{-1}(N^0)$ is nilpotent and consequently $N^0 = \{e\}$, a contradiction. With this the lemma is proved.

Lemma 1.2. *Let H be a connected semi-simple linear Lie group with center Z_H. Let Γ be a closed subgroup of H which is Zariski dense in H and such that $\Gamma \cap Z_H = \{e\}$. Then Γ has no non-trivial nilpotent normal subgroups.*

Proof. Suppose $\Lambda \subset \Gamma$ is a nilpotent normal subgroup of H. Let M be the Zariski closure of Λ. Then M is normalized by Γ. Indeed consider the set $N = \{x \in H: \gamma x \gamma^{-1} \in M, \forall \gamma \in \Gamma\}$. Then N is an algebraic set containing Λ so $M \subset N$. Since Γ is Zariski dense in M we find that G normalizes M. But M is nilpotent so M is discrete hence central. Hence $\Lambda \subset Z_H$ so $\Lambda = \{e\}$.

We now examine the connected component of ρ_0 in $\mathrm{Hom}(\Gamma, U(n, 1))$ using the rigidity theorems of Weil and Mostow. We let $\{\mathscr{R}_i: i = 0, 1, \ldots, l\}$ be the set of components of $\mathrm{Hom}(\Gamma, U(n, 1))$. We number the components so that $\mathscr{R}_0$ is the connected component of ρ_0.

Theorem 1.2. *The connected component of ρ_0 coincides with the set of discrete faithful representations. The group $PU(n, 1) \times \operatorname{Hom}(\Gamma, S^1)$ acts simply transitively on it.*

The theorem is a consequence of the next two lemmas.

Lemma 1.3. *The set of discrete faithful representations of Γ into $U(n, 1)$ is a union of components.*

Proof. We show that the set of discrete faithful representations of Γ into $U(n, 1)$ is open and closed in $\operatorname{Hom}(\Gamma, U(n, 1))$. We observe that since $\rho_0(\Gamma)$ is cocompact we have $H_{2n}(\Gamma, \mathbf{R}) \neq \{0\}$. Consequently any discrete faithful representation of Γ is cocompact. By Weil [16] the set of discrete faithful representations of Γ is open in the strong topology. But by Theorem 1.1 the set of discrete faithful representations of Γ into $U(n, 1)$ is closed and the lemma is proved.

We observe that we have an action of $PU(n, 1) \times \operatorname{Hom}(\Gamma, S^1)$ on $\operatorname{Hom}(\Gamma, U(n, 1))$ by:

$$(g, \chi) \cdot \rho = \operatorname{Ad} g \circ \chi \rho.$$

Here $\chi \rho$ is the representation given by:

$$\chi \rho(\gamma) = \chi(\gamma)\, \rho(\gamma).$$

Lemma 1.4. *$PU(n, 1) \times \operatorname{Hom}(\Gamma, S^1)$ acts simply transitively on the set of discrete faithful representations of Γ in $U(n, 1)$.*

Proof. We have a central extension $S^1 \to U(n, 1) \xrightarrow{\pi} PU(n, 1)$. We observe that if $\rho: \Gamma \to U(n, 1)$ is discrete and faithful then $\rho(\Gamma) \cap S^1 = \{e\}$. Hence $\pi \circ \rho$ is also discrete and faithful. But by the Mostow Rigidity Theorem the group $PU(n, 1)$ acts transitively on the discrete faithful representations of Γ into $PU(n, 1)$. Hence any two elements ρ_1, ρ_2 in $\operatorname{Hom}(\Gamma, U(n, 1))$ satisfy $\pi(\rho_1) = \operatorname{Ad} g \circ \pi(\rho_2)$ for some g in $U(n, 1)$. Hence $\rho_1 = \chi g \circ \rho_2$ for some character χ. This proves that $PU(n, 1) \times \operatorname{Hom}(\Gamma, S^1)$ acts transitively. To prove that the action is free it is sufficient to compute the isotropy at ρ_0. Suppose there exist $h \in PU(n, 1)$ and $\chi \in \operatorname{Hom}(\Gamma, S^1)$ such that:

$$\operatorname{Ad} h \circ \chi \rho_0 = \rho_0.$$

Taking determinants we see that χ has order $n+1$. Letting Γ' be the kernel of χ we find that Γ' is Zariski dense and hence $h = 1$. Hence $\chi = 1$ and the lemma is proved.

Corollary 1.4.1. *The set of discrete faithful representations of Γ into $U(n, 1)$ consists of a single component.*

We now define an embedding j from $U(n, 1)$ into $SU(n+1, 1)$ as follows. Suppose $h \in U(n, 1)$ has matrix (h_{ik}) for $1 \leqq i, k \leqq n+1$. Then we define $j(h) \in SU(n+1, 1)$ by the formula:

$$j(h)\, e_1 = (\det h)^{-1}\, e_1,$$
$$j(h)\, e_k = h_{i-1, k-1}\, e_i \quad \text{for } 2 \leqq i,\ k \leqq n+2.$$

That is:

$$j(h)=\begin{pmatrix}\det h^{-1} & 0\\ 0 & h\end{pmatrix}.$$

Lemma 1.5. *j is a proper map.*

Proof. j is clearly a homeomorphism onto its image and the image of j is described by the linear equations $\{z_{1k}=0\colon 2\leqq k\leqq n+2\}$. Hence the image of j is closed and j is proper.

We let j_* denote the induced map

$$j_*\colon \operatorname{Hom}(\Gamma, U(n,1))\to \operatorname{Hom}(\Gamma, SU(n+1,1)$$

defined by $j_*(\rho)=j\circ\rho$. The image of j_* is again cut out by linear equations and we obtain a lemma.

Lemma 1.6. *j_* is a proper map.*

Corollary 1.6.1. *The decomposition $j_*\operatorname{Hom}(\Gamma, U(n,1))=\bigcup_{i=0}^{l} j_*(\mathscr{R}_i)$ is the decomposition into components.*

We let ρ_0 denote also the composition $j\circ\rho_0$. This composition gives rise to an action of Γ on the unit ball D^{n+1} in $\mathbf{C}^{n+1}$ leaving invariant the linearly embedded n-ball defined by the equation $z_1=0$.

We will be concerned in this paper with the deformations of ρ_0 in $\operatorname{Hom}(\Gamma, G)$. Such deformations would be the analogues for complex hyperbolic space of classical quasi-Fuchsian groups in hyperbolic 3-space.

There are certain obvious deformations of ρ_0. First, there are the trivial deformations. Let g_t be a curve in $SU(n+1,1)$. Then we obtain a deformation ρ_t of ρ_0 by defining $\rho_t=\operatorname{Ad} g_t\circ\rho_0$. Such a deformation is said to be a trivial deformation. We may also deform ρ_0 by characters of Γ. Suppose that the dimension of $H^1(\Gamma,\mathbf{R})$ is greater than zero. Then we may find a curve χ_t in $\operatorname{Hom}(\Gamma, S^1)$ and we obtain a deformation ρ_t of ρ_0 to be denoted $\rho_0*\chi_t$ by the formula:

$$\rho_t(\gamma)\,e_1=\chi_t(\gamma)^{-(n+1)}\,\rho_0(\gamma)\,e_1,$$
$$\rho_t(\gamma)\,e_j=\chi_t(\gamma)\,\rho_0(\gamma)\,e_j \quad \text{for } j\geqq 2.$$

We have $\rho_0*\chi_t=j(\chi_t\,\rho_0)$.

Definition. We say a deformation ρ_t of ρ_0 is Fuchsian if there exists a curve g in G and a curve χ_t in $\operatorname{Hom}(\Gamma, S^1)$ such that:

$$\rho_t=\operatorname{Ad} g_t\circ(\rho_0*\chi_t).$$

Thus the space of Fuchsian deformations of ρ_0 is just the saturation by G of the space $j_*\operatorname{Hom}(\Gamma, U(n,1))$. We wish to analyse this space which will be denoted $R(\rho_0)$. The following general lemma will be very useful to us. Let $\mathbf{P}(W)$ denote the projective space of complex lines in W.

Lemma 1.7. *Let X be compact subset of $\mathbf{P}(W)$. The subset of representations which have a fixed point in X is closed in* $\operatorname{Hom}(\Gamma, G)$.

Proof. We consider the map $F: G^N \times X \to X^{N+1}$ given by:

$$F(g_1, \ldots, g_N, x) = (x, g_1 x, \ldots, g_N x).$$

If Δ denotes the diagonal in X^{N+1}, the condition that ρ has a fixed point in X is that there exists $x \in X$ (the fixed point) such that $(g_1, \ldots, g_N, x) \in F^{-1}(\Delta)$. Let $p_1: G^N \times X \to G^N$ denote the projection. Since the fiber of p_1 is compact, p_1 is a proper and hence a closed map. Hence $p_1(F^{-1}(\Delta))$ is closed and the lemma is proved.

We apply the lemma in two cases. In the first case we let C denote the light cone in W; that is, the set described by:

$$C = \{x \in W: \langle x, x \rangle = 0\}.$$

We let $X = \mathbf{P}C$, the image of C in $\mathbf{P}(W)$.

Corollary 1.7.1. *The set of* $\rho \in \operatorname{Hom}(\Gamma, G)$ *which do not leave a point in* $\mathbf{P}C$ *fixed is open.*

For our second corollary, we observe that $\mathbf{P}(W)$ is separated by $\mathbf{P}C$ into the disjoint union of $\mathbf{P}(W)_+$, the set of positive lines in W and $\mathbf{P}(W)_-$, the set of negative lines in W. Here a line L in W is said to be positive (resp. negative) if $\langle , \rangle | L$ is positive definite (resp. negative definite). By taking $X = \mathbf{P}(W)_- \cup \mathbf{P}C$ we obtain the following corollary.

Corollary 1.7.2. *The set of* $\rho \in \operatorname{Hom}(\Gamma, G)$ *which do not leave a point in* $\mathbf{P}(W)_- \cup \mathbf{P}C$ *fixed is open in* $\operatorname{Hom}(\Gamma, G)$.

We let U' denote this open set. We need a variation on Lemma 1.7 using the Zariski topology.

Lemma 1.8. *Let X be the subset of* $\operatorname{Hom}(\Gamma, \underline{G})$ *consisting of all representations that fix a line. Then X is Zariski closed in* $\operatorname{Hom}(\Gamma, \underline{G})$. *Moreover X is defined over* $\mathbf{R}$.

Proof. We consider the map $F: \underline{G}^N \times \mathbf{P}(W) \to \mathbf{P}(W)^{N+1}$ given by the formula:

$$F(g_1, \ldots, g_N, x) = (x, g_1 x, \ldots, g_N x).$$

Then $X = p_1 F^{-1}(\Delta)$. Since p_1 is proper in the Zariski topology X is Zariski closed.

To see that X is defined over $\mathbf{R}$ we have only to check that if $\rho \in \operatorname{Hom}(\Gamma, G)$ fixes a line then $\sigma(\rho)$ fixes a line. Suppose ρ fixes L. Choose g so that $gL = \mathbf{C}e_1$. Then $\operatorname{Ad} g \circ \rho$ fixes $\mathbf{C}e_1$. Hence $\operatorname{Ad} g \circ \rho$ has the block diagonal form an upper 1 by 1 block with an $n+1$ by $n+1$ block below. But then $(\operatorname{Ad} g \circ \rho)^{*-1}$ has the same block diagonal form and hence $\operatorname{Ad} g^{*-1} \circ (\rho^*)^{-1}$ has the same block diagonal form and fixes $\mathbf{C}e_1$. Consequently $(\rho^*)^{-1}$ fixes $\mathbf{C}g^* e_1$. But then $\sigma(\rho) = \operatorname{Ad} r \circ (\rho^*)^{-1}$ fixes $\mathbf{C}rg^* e_1$. With this the lemma is proved.

Notation. If Y is a subset of $\operatorname{Hom}(\Gamma, G)$ we let $S(Y)$ denote its saturation by G:

$$S(Y) = \{\operatorname{Ad} g \circ y: y \in Y, g \in G\}.$$

The following lemma is obvious since G acts transitively on the positive lines.

Lemma 1.9.

$$X \cap U' = S(\mathrm{Hom}(\Gamma, U(n,1))).$$

We now prove the final lemma of this section.

Lemma 1.10. *The closure of* $S\left(j_*\left(\bigcup_{i=1}^{l} \mathscr{R}_i\right)\right)$ *in* $\mathrm{Hom}(\Gamma, G)$ *does not intersect* $R(\rho_0)$.

Proof. Suppose for the purpose of contradiction that the closure of $S\left(j_*\left(\bigcup_{i=1}^{l} \mathscr{R}_i\right)\right)$ in $\mathrm{Hom}(\Gamma, G)$ intersects $R(\rho_0)$. Then there exist sequences $\{g_n\}$ in G, $\{\rho_n\}$ in $\bigcup_{i=1}^{l} \mathscr{R}_i$ and $\rho \in S(\mathscr{R}_0)$ such that $\lim_{n\to\infty} \mathrm{Ad}\, g_n \circ \rho_n = \rho$. By conjugating by a fixed element of G we may assume ρ has $\mathbf{C}e_1$ as its fixed line. The representation $\mathrm{Ad}\, g_n \circ \rho$ fixes the line $L_n = \mathbf{C} g_n e_1$. By compactness the sequence of fixed lines $\{L_n\}$ in W has a line L of accumulation. By passing to a subsequence we may assume $\lim_{n\to\infty} L_n = L$. But we have $\mathrm{Ad}\, g_n \circ \rho_n L_n = L_n$. Passing to the limit we have $\rho L = L$. But ρ has a unique fixed line. Hence $L = \mathbf{C}e_1$. Now choose a unit vector $v_n \in L_n$ for each n. The sequence $\{L_n\}$ is contained in a closed ball B inside $\mathbf{P}(W)_+$. Hence the sequence $\{v_n\}$ is contained in a compact set - the tautological circle bundle over B. By passing to a subsequence we may assume $\lim_{n\to\infty} v_n = c e_1$ with $c \in S^1$. We redefine v_n to be $c^{-1} v_n$ and we find a sequence of unit vectors $\{v_n\}$ such that v_n is fixed by $\mathrm{Ad}\, g_n \circ \rho_n$ and $\lim_{n\to\infty} v_n = e_1$. We now construct a sequence of elements $\{h_n\}$ in G such that $h_n e_1 = v_n$ and $\lim_{n\to\infty} h_n = \mathrm{id}$. If infinitely many of the v_n's are multiples of e_1 then we are done because $\bigcup_{i=1}^{l} j_*(\mathscr{R}_i)$ is closed by Lemma 1.6. Hence we may assume that none of the v_n's are multiples of e_1. Let

$$a_n = (v_n, e_1), \qquad b_n = \|v_n - \langle v_n, e_1\rangle e_1\|$$

and

$$f_n = \frac{1}{b_n}(v_n - \langle v_n, e_1\rangle e_1).$$

Then $v_n = a_n e_1 + b_n f_n$. Let h_n be the element of G defined by:

$$\begin{aligned} &h_n(e_1) = a_n e_1 + b_n f_n, \\ &h_n(f_n) = \bar{b}_n e_1 + \bar{a}_n f_n, \\ &h_n(u) = u \quad \text{if } \langle u, e_1\rangle = 0 \quad \text{and} \quad \langle u, f_n\rangle = 0. \end{aligned}$$

Then clearly $\{h_n\}$ satisfies the required properties. We now consider the sequence $\{h_n^{-1} g_n \rho_n g_n^{-1} h_n\}$. The elements of this sequence all fix the line $\mathbf{C}e_1$, hence are contained in $j_*\left(\bigcup_{i=1}^{l} \mathscr{R}_i\right)$. But this sequence converges to $\rho \in \mathscr{R}_0$. This is a contradiction since $j_*\left(\bigcup_{i=1}^{l} \mathscr{R}_i\right)$ is closed by Lemma 1.6.

We now define U'' to be the complement of the closure of $S\left(j_*\left(\bigcup_{i=1}^{l} \mathscr{R}_i\right)\right)$ in $\operatorname{Hom}(\Gamma, G)$. We define $U = U' \cap U''$. Then $X \cap U = R(\rho_0)$ and we have proved the following theorem.

Theorem 1.3. *There exists a saturated strongly open subset U of* $\operatorname{Hom}(\Gamma, G)$ *and a saturated real algebraic subset X of* $\operatorname{Hom}(\Gamma, G)$ *such that $X \cap U = R(\rho_0)$.*

The idea of studying the limiting line of the fixed lines of a sequence of Fuchsian representations will allow us to prove the following theorem:

Theorem 1.4. *The set of Fuchsian representations $R(\rho_0)$ is closed in* $\operatorname{Hom}(\Gamma, G)$.

Proof. Suppose $\{\rho_n\}$ is a sequence of representations in $R(\rho_0)$ converging to a representation ρ in $\operatorname{Hom}(\Gamma, G)$. Then ρ is discrete and faithful by Theorem 1.1. Let $\{L_n\}$ to be a sequence of fixed lines as in the previous theorem and L a limit line. If L is a positive line then we are done since ρ fixes L and ρ is discrete and faithful. We may then assume that L is a null-line, for the set of representations fixing a non-negative line is closed by Corollary 1.7.2. Hence ρ is a discrete faithful embedding of Γ into a parabolic subgroup P of G. But P is amenable so Γ is amenable. But this implies Γ is virtually solvable by Tits [14]. With this the lemma is proved.

Thus to prove our main theorem, Theorem 1.2 (Theorem 2 of the introduction), it remains to prove that $R(\rho_0)$ is open in U. We assume for the purpose of contradiction that this is not the case. Hence there exists a sequence of points $\{y_n\}$ in U converging to some $\rho \in R(\rho_0)$. By conjugating by some $g \in G$ we may assume $\rho \in j_* \operatorname{Hom}(\Gamma, G_0)$. We may now apply the Curve Selection Lemma to find a curve ρ_t such that:

(i) $\rho_t \in Y - X$ for $t \neq 0$.
(ii) $\rho_t = \rho$ for $t = 0$.

In the next section we study the Taylor series of ρ_t.

We conclude this section with a theorem concerning $R(\rho_0)$.

Theorem 1.5. *$R(\rho_0)$ is diffeomorphic to $(SU(n+1, 1)/S^1) \times \operatorname{Hom}(\Gamma, S^1)$ where S^1 is embedded in $SU(n+1, 1)$ as the center of $j(U(n, 1))$.*

Proof. If $\rho \in R(\rho_0)$ we let $Z(\rho)$ denote the isotropy subgroup of ρ in G. If $g \in Z(\rho)$ then g stabilizes the invariant line of ρ so $Z(\rho)$ is contained in a conjugate of $j(U(n, 1))$. Applying Lemma 1.4 we obtain the theorem.

2. Construction of a normal cocycle with bounding square

In this section we use the curve ρ_t constructed in the previous section to construct a cocycle $c \in Z^1(\Gamma, \mathfrak{g})$ which is not tangent to the space of Fuchsian deformations, that is $c \notin B^1(\Gamma, \mathfrak{g}) + Z^1(\Gamma, \mathfrak{g}_0)$, and with cup square $[c, c] \in B^2(\Gamma, \mathfrak{g})$. We recall $[c, c]$ is the 2-cocycle defined by:

$$[c, c](\gamma, \eta) = [c(\gamma), \operatorname{Ad} \rho(\gamma) \circ c(\eta)].$$

We then use c to construct a zero for a certain real quadratic form on $H^1(\Gamma, V)$. Here $\mathfrak{g}$ denotes the Lie algebra of G and $\mathfrak{g}_1$, is the orthogonal complement relative to the Killing form of $\mathfrak{g}_0$, the Lie algebra of G_0, in $\mathfrak{g}$. If $c \in H^1(\Gamma, \mathfrak{g})$, the class of $[c, c]$ is the first obstruction to finding a curve ρ_t tangent to c.

We will need to generalize the material in Johnson-Millson [6], Sect. 2 to the case in which the tangent vector of a deformation vanishes at $t=0$. We will also need to break up the deformation into a block diagonal and off diagonal part and generalize the two main lemmas, Lemma 2.1 and Lemma 2.3 of [6] to the leading coefficient of the off diagonal part.

We begin with an easy lemma as a bridge to the more detailed generalizations we will require.

Definition. Let X be an affine variety in $\mathbf{R}^m$ and $\alpha: (-\varepsilon, \varepsilon) \to X$ be a real analytic curve such that $\alpha(0) = x$. Let $\alpha(t) = \sum_{k=0}^{\infty} \alpha_k(0)\, t^k$ be the Taylor series for α about $t=0$. We define the leading coefficient of α to be α_n if $n>0$, $\alpha_n \neq 0$ and $\alpha_m = 0$ for $0<m<n$.

In other words α_n is the first non-zero term after the constant term. We observe that $\alpha_n \in \mathbf{R}^m$.

Lemma 2.1. *The leading coefficient α_n of α is tangent to X at x.*

Proof. Let f be a polynomial function vanishing on X. Then $f(\alpha(t)) = 0$ for all t. We may assume $x=0$. We expand $f(\alpha(t))$ around $t=0$ and find that $f(\alpha(t)) \equiv \mathrm{d}f_0(\alpha_n)\, t^n \bmod t^{n+1}$. Hence $\mathrm{d}f_0$ annihilates α_n and the lemma is proved.

Corollary. *Let $G \subset GL(k, \mathbf{C})$ be a closed subgroup. Suppose ρ_t is a curve in* $\operatorname{Hom}(\Gamma, G)$ *and ρ_n is the leading term of ρ_t. Then the function $c: \Gamma \to \mathrm{gl}(k, \mathbf{C})$ defined by:*

$$c(\gamma) = \rho_n(\gamma)\, \rho_0(\gamma)^{-1}$$

is a cocycle on Γ with values in $\mathfrak{g}$. Furthermore if ρ_t is a trivial deformation then c is a coboundary.

Let U be a neighborhood of O in $\mathfrak{g}$ such that the restriction of $\exp: \mathfrak{g} \to G$ to U is injective. Let $F \subset \Gamma$ be an arbitrary finite set. Then for sufficiently small $\varepsilon > 0$ we have $\rho_t(\gamma)\, \rho_0(\gamma)^{-1} \in \exp(U)$ for all $\gamma \in F$ and $|t| < \varepsilon$. Thus we may write for $\gamma \in F$:

$$\rho_t(\gamma)\, \rho_0(\gamma)^{-1} = \exp u_t(\gamma)$$

where $u_t(\gamma)$ is curve in $\mathfrak{g}$ with $u_0(\gamma) = 0$.

We write out the Taylor series for $u_t(\gamma)$ with $\gamma \in \Gamma$ as:

$$u_t(\gamma) = \sum_{k=0}^{\infty} u_k(\gamma)\, t^k.$$

Substituting into the power series for exp we find that $u_k(\gamma) = 0$ for $k < n$ and $u_n(\gamma) = c(\gamma)$ (where $\rho_n(\gamma)$ is the leading coefficient of $\rho_t(\gamma)$ and $c(\gamma) = \rho_n(\gamma)\, \rho_0(\gamma)^{-1}$) as above).

We next observe that if F generates Γ then there exists k so that $u_k(\gamma)\notin \mathfrak{g}_0$. Otherwise $u_t(\gamma)\in\mathfrak{g}_0$ for all t and $\gamma\in F$ and hence $\rho_t(\gamma)=\exp u_t(\gamma)\,\rho_0(\gamma)$ is an element of G_0 for all t and $\gamma\in F$. Since F generates Γ this is a contradiction.

We observe that as a consequence of the identity $\rho_t(\gamma\eta)=\rho_t(\gamma)\,\rho_t(\eta)$ we obtain:

$$\exp u_t(\gamma\eta)=\exp u_t(\gamma)\exp \operatorname{Ad}\rho_0(\gamma)\,u_t(\eta).$$

We now derive some consequences of the Campbell-Baker-Hausdorff formula. We have

$$\exp u_t(\gamma)\exp\operatorname{Ad}\rho_0(\gamma)\,u_t(\eta)=\exp\left(\sum_{n=0}^{\infty} C_n(u_t(\gamma),\operatorname{Ad}\rho_0(\gamma)\,u_t(\eta))\right)$$

where C_n is the nth Campbell-Baker-Hausdorff polynomial, Bourbaki [2], p. 55. We obtain an equality of power series:

$$\begin{aligned}\sum_{n=1}^{\infty} u_n(\gamma\eta)\,t^n=\sum_{n=1}^{\infty} u_n(\gamma)\,t^n&+\sum_{n=1}^{\infty}\operatorname{Ad}\rho_0(\gamma)\,u_n(\eta)\,t^n\\ &+\frac{1}{2}\sum_{n=1}^{\infty}\sum_{i+j=n}[u_i(\gamma),\operatorname{Ad}\rho(\gamma)\,u_j(\eta)]\,t^n\\ &+\sum_{n=3}^{\infty} C_n\left(\sum_{i=1}^{\infty} u_i(\gamma)\,t^i,\sum_{i=1}^{\infty}\operatorname{Ad}\rho_0(\gamma)\,u_i(\eta)\,t^i\right).\end{aligned}$$

Bringing the first two terms to the left-hand side and equating coefficients of t^n we obtain a multilinear function N_n on the Lie algebra such that:

$$\delta u_n(\gamma,\eta)=N_n(u_1(\gamma),\ldots,u_{n-1}(\gamma);\operatorname{Ad}\rho_0(\gamma)\,u_1(\eta),\ldots,\operatorname{Ad}\rho_0(\gamma)\,u_{n-1}(\eta)).$$

We observe that N_n is a sum of monomials which are brackets of the $u_i(\gamma)$'s and the $\operatorname{Ad}\rho_0(\gamma)\,u_j(\eta)$'s. In particular if u_i takes values in an abelian Lie algebra for $i<n$ then u_n is a cocycle.

We now observe that $\mathfrak{g}$ is $\mathbf{Z}/2$ graded by the decomposition described earlier. That is, we have a decomposition $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ with the bracket relations:

(i) $[\mathfrak{g}_0,\mathfrak{g}_0]\subset\mathfrak{g}_0$.
(ii) $[\mathfrak{g}_0,\mathfrak{g}_1]\subset\mathfrak{g}_1$.
(iii) $[\mathfrak{g}_1,\mathfrak{g}_1]\subset\mathfrak{g}_0$.

In particular any word $[[[x_1,x_2],x_3],\ldots,x_n]$ in $\mathfrak{g}$ with the x_j's in either $\mathfrak{g}_0$ or $\mathfrak{g}_1$ will be in $\mathfrak{g}_0$ if an even number of the x_j's are in $\mathfrak{g}_1$ and will be in $\mathfrak{g}_1$ if an odd number of the x_j's are in $\mathfrak{g}_1$. Corresponding to the decomposition of $\mathfrak{g}$ we have a decomposition of cochains $u:\Gamma\to\mathfrak{g}$ as $u=w+v$ with $w:\Gamma\to\mathfrak{g}_0$ and $v:\Gamma\to\mathfrak{g}_1$.

We also assume we have a decomposition $\mathfrak{g}_0=\mathfrak{g}_0'\oplus\mathfrak{z}_0$ of $\mathfrak{g}_0$ as the direct sum of two ideals such that the map $H^1(\Gamma,\mathfrak{z}_0)\to H^1(\Gamma,\mathfrak{g}_0)$ is an isomorphism. If $x\in\mathfrak{g}_0$ we let x' denote the projection of x on $\mathfrak{g}_0'$. Clearly this projection is a Lie algebra homomorphism and commutes with the group coboundary δ.

We observe that this last assumption is satisfied when $\mathfrak{g}$ is the Lie algebra of $SU(n+1, 1)$, $\mathfrak{g}_0$ is the Lie algebra of $U(n, 1)$ and $\mathfrak{z}_0$ is the center of $\mathfrak{g}_0$. The preceding statement is a consequence of Weil's vanishing theorem, see [12], VII-5, recall that we are assuming $n \geqq 2$. However for later applications we continue in the more general situation of arbitrary Γ and arbitrary $\mathfrak{g}$ subject to the hypotheses of the previous two paragraphs.

We define the *leading coefficient* mod $\mathfrak{g}_0$ of a deformation ρ_t to be v_m where u_m is the first term such that $u_m(\gamma) \notin \mathfrak{g}_0$ for some $\gamma \in \Gamma$ and $u_m = w_m + v_m$ is the above decomposition. Here u_m is the m-th Taylor coefficient of the curve u_t associated to ρ_t as above.

Lemma 2.2. *The leading coefficient* mod $\mathfrak{g}_0$ *is a cocycle.*

Proof. To see that v_m is a cocycle compare $\mathfrak{g}_1$ parts of degree t^m on the two sides of the equation:

$$\exp u_t(\gamma\eta) = \exp u_t(\gamma) \exp(\rho_0(\gamma)\, u_t(\eta)\, \rho_0(\gamma)^{-1}).$$

We have seen that δu_m is a sum of iterated brackets of lower order terms. But these lower order terms are all in $\mathfrak{g}_0$. Hence $\delta(w_m + v_m)$ is in $\mathfrak{g}_0$ and consequently $\delta v_m = 0$. With this the lemma is proved.

We say a deformation ρ_t is *normalized* if all the coefficients u_k of $\log(\rho_t \rho_0^{-1})$ take values in $\mathfrak{z}_0$ for $k \leqq m$. Here m is such that v_m is the leading coefficient mod $\mathfrak{g}_0$.

Lemma 2.3. *If ρ_t is any deformation of ρ_0 then there exists an analytic curve h_t in G_0 with $h_0 = e$ such that* $\operatorname{Ad} h_t \circ \rho_t$ *is normalized.*

Proof. We observe that by assumption $H^1(\Gamma, \mathfrak{z}_0)$ maps onto $H^1(\Gamma, \mathfrak{g}_0)$. Hence if $u \in Z^1(\Gamma, \mathfrak{g}_0)$ then there exists $\eta \in Z^1(\Gamma, \mathfrak{z}_0)$ and $b \in B^1(\Gamma, \mathfrak{g}_0)$ such that $u = \eta + b$.

We will prove the lemma by induction on the degree. We first normalize the leading coefficient u_n. We choose η_n and b_n as above such that $u_n = \eta_n + b_n$. We choose a curve h_t in G_0 such that if $x = \dot{h}$ then $b_n(\gamma) = \operatorname{Ad} \rho_0(\gamma)\, x - x$. Then the leading coefficient of $\operatorname{Ad} h_{t^n} \circ \rho_t$ is the coefficient of t^n in the polynomial $(1 + x t^n)(\rho_0 + u_n t^n \rho_0)(1 - x t^n)$, that is $x \rho_0 - \rho_0 x + u_n \rho_0$. The corresponding cocycle is η_n and we have normalized the leading term.

Let us suppose that the first k terms are normalized. We wish to normalize u_{k+1}. By the remarks preceding this lemma we have $\delta u_{k+1} = N_{k+1}(u_1, \ldots, u_k)$. But $u_1, \ldots, u_k$ take values in the Lie algebra $\mathfrak{z}_0$ so $N_{k+1}(u_1, \ldots, u_k)' = 0$ and u'_{k+1} is a cocycle. Hence there exist b_{k+1} and η_{k+1} with $b_{k+1} \in B^1(\Gamma, \mathfrak{g}_0)$ and $\eta_{k+1} \in C^1(\Gamma, \mathfrak{z}_0)$ such that $u_{k+1} = \eta_{k+1} + b_{k+1}$. We now construct a curve h_t as before so that $h_t \equiv 1 + x t^{k+1} \bmod t^{k+2}$ and $b_{k+1}(\gamma) = \operatorname{Ad} \rho(\gamma)\, x - x$. We claim that the curve $\rho'_t = \operatorname{Ad} h_t \circ \rho_t$ has its first $k+1$ terms normalized. Indeed we have:

$$\begin{aligned} \rho'_t &\equiv (1 + x t^{k+1})(\exp u_t)\, \rho_0 (1 - x t^{k+1}) \bmod t^{k+2} \\ &\equiv (\exp u_t)\, \rho_0 + (x \rho_0 - \rho_0 x)\, t^{k+1} \bmod t^{k+2}. \end{aligned}$$

Hence:

$$\rho'_t \rho_0^{-1} \equiv \exp u_t - b_{k+1}\, t^{k+1} \bmod t^{k+2}.$$

We see that $\log(\rho_t' \rho_0^{-1})$ has the same coefficients as u_t up to and including the terms of order k. The coefficient of t^{k+1} is $u_{k+1} - b_{k+1} = \eta_{k+1}$ and the lemma is proved.

We are now ready to prove our main lemma. We now assume $\mathscr{z}_0$ is abelian, hence central in $\mathscr{g}_0$.

Lemma 2.4. *If ρ_t is a normalized deformation then we have:*

$$\delta w_{2m} = [v_m, v_m].$$

Proof. We examine the Campbell-Baker-Hausdorff formula more carefully to obtain:

$$\begin{aligned}\delta u_{2m}(\gamma, \eta) = &\frac{1}{2} \sum_{k=1}^{2m-1} [u_k(\gamma), \operatorname{Ad} \rho_0(\gamma)\, u_{2m-k}(\eta)] \\ &+ M_{2m}(u_1(\gamma), \ldots, u_{2m-1}(\gamma); \operatorname{Ad} \rho_0(\gamma)\, u_1(\eta), \ldots, \operatorname{Ad} \rho_0(\gamma)\, u_{2m-1}(\eta)).\end{aligned}$$

Here M_{2m} is the sum of all monomials in N_{2m} involving at least three brackets.

We now decompose each u_k according to $u_k = w_k + v_k$. Hence

$$\operatorname{Ad} \rho_0(\gamma)\, u_k = \operatorname{Ad} \rho_0(\gamma)\, u_k + \operatorname{Ad} \rho_0(\gamma)\, v_k$$

for all $\gamma \in \Gamma$. We now expand the first term on the right-hand side of the above equation.

$$\begin{aligned}\frac{1}{2} \sum_{k=1}^{2m-1} [u_k(\gamma), \operatorname{Ad} \rho_0(\gamma)\, u_{2m-k}(\eta)] = &\frac{1}{2} \sum_{k=1}^{2m-1} [w_k(\gamma), \operatorname{Ad} \rho_0(\gamma)\, w_{2m-k}(\eta)] \\ &+ \frac{1}{2} \sum_{k=1}^{m} [w_k(\eta), \operatorname{Ad} \rho_0(\gamma)\, v_{2m-k}(\eta)] \\ &+ \frac{1}{2} \sum_{k=m}^{2m} [v_k(\gamma), \operatorname{Ad} \rho_0(\gamma)\, w_{2m-k}(\eta)] \\ &+ \frac{1}{2} [v_m(\gamma), \operatorname{Ad} \rho_0(\gamma)\, v_m(\eta)].\end{aligned}$$

We see that the two middle terms on the right are annihilated by the projection on $\mathscr{g}_0$. Also the first term is zero for in each of the brackets one of the terms is in $\mathscr{z}_0$ and the other in $\mathscr{g}_0$. We now analyze M_{2m}.

We expand M_{2m} by the total number of brackets to obtain:

$$\begin{aligned}&M_{2m}(w_i(\gamma), v_i(\gamma); \operatorname{Ad} \rho_0(\gamma)\, w_i(\eta), \operatorname{Ad} \rho_0(\gamma)\, v_i(\eta)) \\ &\quad = \sum_{j=3}^{2m} A_j(w_i(\gamma), v_i(\gamma); \operatorname{Ad} \rho_0(\gamma)\, w_i(\eta), \operatorname{Ad} \rho_0(\gamma)\, v_i(\eta)).\end{aligned}$$

Here A_j is a sum of monomials each of which is a j-fold bracket. Since we have projected onto $\mathscr{g}_0$ each monomial is an element of $\mathscr{g}_0$. The total t-degree is $2m$. We claim that the A_j's involve only the w's. Indeed any monomial occurring in the A_j's is in $\mathscr{g}_0$ and hence must involve an even number of v's by the discussion preceding Lemma 2.1. But if any v's occur then the monomial will have t-degree larger than or equal to $n+2m$. Thus, the A_j's do not depend on the v's.

Since $j \geqq 3$ it is clear that at least one w_k or $\mathrm{Ad}\,\rho_0(\gamma)\,w_k$ occurring in a monomial must satisfy $k<m$. Say w_{k_0} is such a term. Then w_{k_0} takes values in $\mathfrak{z}_0$. But w_{k_0} occurs bracketed with a monomial in the w's which take values in $\mathfrak{g}_0$. The resulting bracket with w_{k_0} must be zero since w_{k_0} takes values in $\mathfrak{z}_0$. With this the lemma is proved.

We now modify the curve ρ_t so that the leading coefficient mod $\mathfrak{g}_0$ is not a coboundary. We have the following lemma.

Lemma 2.5. *If ρ_t is any deformation of ρ_0 then there exists an analytic curve g_t in G with $g_0=e$ such that either* $\mathrm{Ad}\,g_t \circ \rho_t$ *is normalized with leading coefficient* mod $\mathfrak{g}_0$ *not a coboundary or* $\mathrm{Ad}\,g_t \circ \rho_t$ *is contained in* $\mathrm{Hom}(\Gamma, G_0)$.

Proof. By Lemma 2.3 we may assume that ρ_t is normalized. Suppose v_{m_1} is the leading coefficient mod $\mathfrak{g}_0$ of ρ_t. Suppose v_{m_1} is a coboundary. Then there exists $x \in \mathfrak{g}$ such that $v_{m_1}(\gamma)=\mathrm{Ad}\,\rho_0(\gamma)\,x-x$. Let $g_{m_1,t}$ be the one parameter group in G with $\dot{g}_{m_1}=x$. We consider the curve ρ'_t given by $\rho'_t=\mathrm{Ad}\,g_{m_1,t^{m_1}} \circ \rho_t$. We define a curve u'_t with values in $\mathfrak{g}$ as before with $u'_t=\log(\rho'_t \circ \rho_0^{-1})$ and write $u'_t(\gamma)=\sum_{k=1}^{\infty} u'_k(\gamma)\,t^k$. Then a calculation similar to that of Lemma 2.3 shows that $u'_k=u_k$ for $k<m_1$ and $u'_{m_1}=u_{m_1}-v_{m_1}$. We see then that the curve ρ'_t has leading coefficient mod $\mathfrak{g}_0$ equal to v_{m_2} with $m_2>m_1$. We normalize ρ'_t and repeat the above process. Either we arrive at a deformation $\mathrm{Ad}\,g_t \circ \rho_t$ with leading coefficient mod $\mathfrak{g}_0$ which is not a coboundary or we obtain a *formal* curve $g'_t=\prod_{k=1}^{\infty} g_{m_k,t^{m_k}}$ which satisfies the analytic equations given by $\mathrm{Ad}\,g'_t \circ \rho_t \in \mathrm{Hom}(\Gamma, Z_0)$ where Z_0 is the analytic subgroup of G_0 corresponding to $\mathfrak{z}_0$. But then by Artin [18] there exists an *analytic* curve g_t such that $\mathrm{Ad}\,g_t \circ \rho_t \in \mathrm{Hom}(\Gamma, Z_0)$. With this the lemma is proved.

We now examine more closely the cup-product of the beginning of this section. The cup-product of cochains gives rise to a map:

$$H^1(\Gamma, \mathfrak{g}) \times H^1(\Gamma, \mathfrak{g}) \xrightarrow{\otimes} H^2(\Gamma, \mathfrak{g} \otimes \mathfrak{g})$$

given by

$$c \otimes c(\gamma, \eta)=c(\gamma) \otimes \mathrm{Ad}\,\rho(\gamma) \cdot c(\eta).$$

But the Lie bracket gives a Γ-module map $\mathfrak{g} \otimes \mathfrak{g}$ into $\mathfrak{g}$ inducing a map from $H^2(\Gamma, \mathfrak{g} \otimes \mathfrak{g})$ to $H^2(\Gamma, \mathfrak{g})$. The composition of this map with the cup-product gives rise to the map

$$H^1(\Gamma, \mathfrak{g}) \times H^1(\Gamma, \mathfrak{g}) \xrightarrow{[\,,\,]} H^2(\Gamma, \mathfrak{g})$$

described in the beginning of this section given by:

$$[\alpha, \alpha](\gamma, \eta)=[\alpha(\gamma)\ \mathrm{Ad}\,\rho(\gamma) \cdot \alpha(\eta)].$$

We observe that $[\,,\,]$ induces a map

$$H^1(\Gamma, \mathfrak{g}_1) \otimes H^1(\Gamma, \mathfrak{g}_1) \to H^2(\Gamma, \mathfrak{g}_0).$$

We may apply the considerations of this section to the curve ρ_t constructed in Sect. 1. We let $c=v_m$ where v_m is the leading coefficient of $\rho_t \bmod \mathfrak{g}_0$. We have observed that since ρ_t is not contained in $\mathrm{Hom}(\Gamma, G_0)$ there exists m so that $v_m \neq 0$. Moreover, since ρ_t is not contained in $S(\mathrm{Hom}(\Gamma, G_0))$ for $t>0$, there is no curve g_t such that $\mathrm{Ad}\, g_t \circ \rho_t \in \mathrm{Hom}(\Gamma, G_0)$. By Lemma 2.5, we may assume that v_m is not a coboundary. But $[v_m, v_m]$ is a coboundary by Lemma 2.4. We let Γ act on $\mathfrak{g}$ by ρ_0. We observe that $\rho_0(\Gamma) \subset j(U(n,1))$ and consequently we may identify Γ with $j^{-1} \circ \rho_0(\Gamma)$ a cocompact discrete subgroup of $U(n,1)$. We obtain the following theorem.

Theorem 2.1. *Suppose $R(\rho_0)$ is not open in* $\mathrm{Hom}(\Gamma, G)$. *Then there exists a realization of Γ as a cocompact torsion-free discrete subgroup of $U(n,1)$ and a non-zero class $c \in H^1(\Gamma, \mathfrak{g}_1)$ such that $[c,c]$ is zero in $H^2(\Gamma, \mathfrak{g}_0)$.*

We now observe that the mapping trace: $\mathfrak{g}_0 \to \mathbf{C}$ induces a map $H^2(\Gamma, \mathfrak{g}_0) \to H^2(\Gamma, \mathbf{C})$. Moreover, a calculation shows that trace takes pure imaginary values on $\mathfrak{g}_0$. Multiplying by $i/2$ we obtain a map from $H^2(\Gamma, \mathfrak{g}_0)$ to $H^2(\Gamma, \mathbf{R})$. By composition we obtain a map:

$$\Psi: H^1(\Gamma, \mathfrak{g}_1) \times H^1(\Gamma, \mathfrak{g}_1) \to H^2(\Gamma, \mathbf{R}).$$

We now use some considerations from differential geometry to get a real quadratic form on $H^1(\Gamma, \mathfrak{g}_1)$. We may identify the homology and cohomology groups of Γ with real coefficients with the corresponding homology and cohomology groups of the compact manifold $M=\Gamma \backslash D$. But M is a projective variety and consequently has a distinguished class Z in $H_2(M, \mathbf{Z})$. This class is obtained as follows. Let $H \subset M$ be the hyperplane section class of the projective embedding given by applying the Kodaira Embedding Theorem to the Kahler form ω normalized to have integral periods. ω is (a multiple of) the imaginary part of the Bergmann metric. We observe that $H \in H_{2n-2}(M, \mathbf{Z})$ and ω is the Poincare dual of H. Then Z, the $(n-1)$-fold self-intersection class of H, is a non-zero element of $H_2(M, \mathbf{Z})$ which is Poincare dual to ω^{n-1}.

We now define a quadratic form Q on $H^1(\Gamma, \mathfrak{g}_1)$ by:

$$Q(c) = \Psi(c,c) \cdot Z.$$

Here we use $\cdot$ to denote the Kronecker index.

We now observe that there is a vector space isomorphism from V to $\mathfrak{g}_1$ given by sending $(z_1, z_2, \ldots, z_{n+1})$ to the $n+2$ by $n+2$ matrix with first column $(0, z_1, z_2, \ldots, z_{n+1})$ and first row $(0, -\bar{z}_1, \ldots, -\bar{z}_n, \bar{z}_{n+1})$ and all other entries zero. The action of $\rho_0(\Gamma)$ on $\mathfrak{g}_1$ corresponds to the standard action of $\Gamma \subset U(n,1)$ on $\mathbf{C}^{n+1}$ twisted by the restriction of the determinant. We then have the following consequence of Theorem 2.1.

Theorem 2.1 (bis). *Suppose $R(\rho_0)$ is not open in* $\mathrm{Hom}(\Gamma, G)$. *Then there exists a non-zero class c in $H^1(\Gamma, V)$ which is a zero of the real quadratic form Q defined by the composition:*

$$H^1(\Gamma, V) \times H^1(\Gamma, V) \to H^2(\Gamma, \mathbf{R}) \to \mathbf{R}.$$

Here the first arrow is the cup-product induced by the real bilinear map given by the imaginary part of $\langle\, , \rangle$. The second arrow is evaluation on the class Z.

3. Some hermitian linear algebra

The purpose of the last two sections of our paper is to prove that the quadratic form Q on $H^1(\Gamma, V)$ defined at the end of the last section is in fact positive definite. Thus the assumption that $R(\rho_0)$ is not open in $\mathrm{Hom}(\Gamma, G)$ leads to a contradiction and our main theorem is proved.

In this section we prove a definiteness theorem for a certain quadratic form. This theorem will imply that certain integrands considered in Sect. 4 are pointwise either positive or negative. This will in turn imply that Q will have a fixed sign on each of the four Hodge groups associated to $H^1(\Gamma, V)$. We will then invoke a theorem of Matsushima-Murakami which implies that the two Hodge groups on which Q is negative definite are in fact zero.

Let U be a real vector space with a complex structure J and an inner product $(\,,)$ such that for all u_1, u_2 in U we have:

$$(Ju_1, Ju_2) = (u_1, u_2).$$

The J acts on U^*, the dual space of U, by

$$J \cdot \alpha(u) = -\alpha(Ju).$$

We define $\omega \in \Lambda^2 U^*$, the Kahler form associated to $(\,,)$ and J by:

$$\omega(u_1, u_2) = (Ju_1, u_2).$$

We can choose an orthonormal basis $\{q_j, p_j : j = 1, 2, \ldots, n\}$ for U^* such that for all j:

$$Jq_j = p_j$$
$$Jp_j = -q_j.$$

We find that the Kahler form ω is given by:

$$\omega = \sum_{j=1}^{n} q_j \wedge p_j.$$

We now consider elements of $U^* \otimes_{\mathbf{R}} \mathbf{C}$. We define an element $\alpha \in U^* \otimes_{\mathbf{R}} \mathbf{C}$ to be of type $(1, 0)$ if α satisfies for all $u \in U$:

$$\alpha(Ju) = i\alpha(u) \qquad \text{or equivalently} \quad J\alpha = -i\alpha.$$

We denote the subspace of $U^* \otimes \mathbf{C}$ (we drop the subscript $\mathbf{R}$ on the tensor products henceforth) of type $(1, 0)$ linear functionals by $(U^* \otimes \mathbf{C})^{(1,0)}$.

We define an element β of $U^* \otimes \mathbf{C}$ to be of type $(0, 1)$ if β satisfies for all $u \in U$:

$$\beta(Ju) = -i\beta(u) \qquad \text{or equivalently} \quad J\beta = i\beta.$$

Now if $\alpha \in U^* \otimes \mathbf{C}$ then we define its complex conjugate $\bar{\alpha} \in U^* \otimes \mathbf{C}$ by:

$$\bar{\alpha}(v) = \overline{\alpha(v)}.$$

Clearly the operation of complex conjugation interchanges $(U^* \otimes \mathbf{C})^{(1,0)}$ and $(U^* \otimes \mathbf{C})^{(0,1)}$. Also we note that $\{p_j + i q_j : j = 1, 2, \ldots, n\}$ is a basis for $(U^* \otimes \mathbf{C})^{(1,0)}$ considered as a complex vector space and $\{p_j - i q_j : j = 1, 2, \ldots, n\}$ is a basis for $(U^* \otimes \mathbf{C})^{(0,1)}$ considered as a complex vector space.

We define $\tau \in \Lambda^2 U^*$ to be of type (1, 1) if $\tau(J u_1, J u_2) = \tau(u_1, u_2)$. We observe that ω is of type (1, 1).

We will also need to consider $U^* \otimes_{\mathbf{R}} V$ with V a complex vector space equipped with a non-singular sesquilinear form $\langle\, , \rangle$. We let τ denote the imaginary part of $\langle\, , \rangle$. Since V is a complex vector space we may define type (1, 0) forms, type (0, 1) forms and complex conjugation using the same formulas as above, noting the difference that $\alpha(u)$ is now an element of V. Observe that if $\alpha \in U^*$ and $v \in V$ we have the dyad $\alpha \otimes v \in U^* \otimes V$ defined by:

$$\alpha \otimes v(u) = \alpha(u)\, v.$$

We now define a real bilinear map B on $U^* \otimes V$ with values in the type (1, 1) elements of $\Lambda^2 U^*$ by the formula:

$$B(\alpha, \beta)(u_1, u_2) = \tau(\alpha(u_1), \beta(u_2)) - \tau(\alpha(u_2), \beta(u_1)).$$

We note that this formula is the same as:

$$B(\alpha, \beta)(u_1, u_2) = \operatorname{Im}(\langle \alpha(u_1), \beta(u_2) \rangle - \langle \alpha(u_2), \beta(u_1) \rangle).$$

It is important to observe that B *is symmetric*, that is, $B(\alpha, \beta) = B(\beta, \alpha)$.

Lemma 3.1. *Suppose α, β are in $U^* \otimes \mathbf{C}$ and v, v' are in V. Then:*

(i) *If v and v' are orthogonal we have:*

$$B(\alpha \otimes v, \beta \otimes v') = 0.$$

(ii) $B(\alpha \otimes v, \alpha \otimes v) = \langle v, v \rangle \operatorname{Im} \alpha \wedge \bar{\alpha}$.

Here if α, β are in $U^* \otimes \mathbf{C}$ then $\operatorname{Im}(\alpha \wedge \beta)$ is the element of $\Lambda^2 U^*$ whose value at (u_1, u_2) is the imaginary part of the complex number $\alpha(u_1)\,\beta(u_2) - \alpha(u_2)\,\beta(u_1)$.

Proof. From the defining formula for B we have for $u_1, u_2 \in U$:

$$\begin{aligned} B(\alpha \otimes v, \beta \otimes v')(u_1, u_2) &= \operatorname{Im}(\langle \alpha(u_1)\, v, \beta(u_2)\, v' \rangle - \langle \alpha(u_2)\, v, \beta(u_1)\, v' \rangle) \\ &= \operatorname{Im}(\alpha(u_1)\, \bar{\beta}(u_2) \langle v, v' \rangle - \alpha(u_2)\, \bar{\beta}(u_1) \langle v, v' \rangle). \end{aligned}$$

The lemma is now obvious.

We now define a real-valued symmetric form $((\, ,))$ on $U^* \otimes V$ by:

$$((\alpha, \beta)) = (B(\alpha, \beta), \omega).$$

Here the inner product on the right is the inner product on $\Lambda^2 U^*$ induced by $(\, ,)$.

Lemma 3.2. (i) *Suppose* $\alpha \in (U^* \otimes \mathbf{C})^{(1,0)}$, *then:*

$$(\operatorname{Im} \alpha \wedge \bar{\alpha}, \omega) = -\|\alpha\|^2.$$

(ii) *Suppose* $\alpha \in (U^* \otimes \mathbf{C})^{(0,1)}$, *then:*

$$(\operatorname{Im} \alpha \wedge \bar{\alpha}, \omega) = \|\alpha\|^2.$$

Proof. The second statement follows from the first since $\alpha \to \bar{\alpha}$ interchanges $(U^* \otimes \mathbf{C})^{(1,0)}$ and $(U^* \otimes \mathbf{C})^{(0,1)}$. To prove the first statement note that both sides are bilinear functions on the *real* vector space $(U^* \otimes \mathbf{C})^{(1,0)}$. Hence it is sufficient to check the identity on the real basis

$$\{q_j + ip_j, i(q_j + ip_j): 1 \leqq j \leqq n\}.$$

But observing that both sides are unchanged upon replacing α by $i\alpha$ we see that is sufficient to prove for $1 \leqq j \leqq n$:

$$\operatorname{Im}((q_j + ip_j) \wedge (q_j - ip_j), \omega) = -2.$$

But this is clear for

$$(q_j + ip_j) \wedge (q_j - ip_j) = -2iq_j \wedge p_j.$$

With this the lemma is proved.

Let $V = V^+ \otimes V^-$ be a splitting of V into the orthogonal sum of a positive definite subspace and a negative definite subspace. We obtain a corresponding decomposition:

$$U^* \otimes V = (U^* \otimes V^+)^{(1,0)} \oplus (U^* \otimes V^-)^{(1,0)} \oplus (U^* \otimes V^+)^{(0,1)} \oplus (U^* \otimes V^-)^{(0,1)}.$$

Theorem 3.1. (i) *The summands in the above decomposition are orthogonal for* $((\ ,\ ))$.

(ii) $((\ ,\ ))$ *is negative definite on the first and last summands and positive definite on the other two summands.*

Proof. The first statement follows from Lemma 3.1. It is sufficient to prove all dyads from one space are orthogonal to all dyads from another. Let us consider $U^* \otimes V^+)^{(1,0)}$. Clearly it is orthogonal to the second and fourth summands by Lemma 3.1. But it is orthogonal to the third summand since if α has type $(1,0)$ and β has type $(0,1)$ then $\operatorname{Im} \alpha \wedge \bar{\beta}$ will have no type $(1,1)$ component and will consequently be orthogonal to ω. The orthogonality of the other summands may be verified in a similar fashion. We now prove (ii).

Let $\{v'_j: 1 \leqq j \leqq r\}$ be an orthonormal basis for V^+ and $\{v''_j: 1 \leqq j \leqq s\}$ be an orthogonal basis for V^- satisfying $\langle v''_j, v''_j \rangle = -1$ for $j = 1, 2, \ldots, s$. Let $\alpha \in (U^* \otimes V^-)^{(1,0)}$. Then α may be written as $\alpha = \sum_{j=1}^{s} \alpha_j \otimes v''_j$ where $\alpha_j \in (U^* \otimes \mathbf{C})^{(1,0)}$. Then we have (by Lemma 3.1):

$$B(\alpha \wedge \alpha) = \sum_j \langle v''_j, v''_j \rangle \operatorname{Im}(\alpha_j \wedge \bar{\alpha}_j).$$

Hence we obtain:

$$((\alpha, \alpha)) = (B(\alpha \wedge \alpha), \omega) = \sum_j \langle v_j'', v_j''\rangle (\operatorname{Im}(\alpha_j \wedge \bar{\alpha}_j), \omega) = -\sum_j \langle v_j'', v_j''\rangle \|\alpha_j\|^2.$$

The other statements can be proved in a similar fashion, all dyads from one space are orthogonal to all dyads from another. Let us consider $(U^* \otimes V^+)^{(1,0)}$. Clearly it is orthogonal to the second and fourth summands by Lemma 3.1. But it is orthogonal to the third summand since if α has type $(1,0)$ and β has type $(0,1)$ then $\operatorname{Im} \alpha \wedge \bar{\beta}$ will have no type $(1,1)$ component and will consequently be orthogonal to ω. The orthogonality of the other summands may be verified in a similar fashion. We now prove (ii).

Let $\{v_j': 1 \leqq j \leqq r\}$ be an orthonormal basis for V^+ and $\{v_j'': 1 \leqq j \leqq s\}$ be an orthogonal basis for V^- satisfying $\langle v_j'', v_j''\rangle = -1$ for $j = 1, 2, \ldots, s$. Let $\alpha \in (U^* \otimes V^-)^{(1,0)}$. Then α may be written as $\alpha = \sum_{j=1}^{s} \alpha_j \otimes v_j''$ where $\alpha_j \in (U^* \otimes \mathbf{C})^{(1,0)}$. Then we have (by Lemma 3.1):

$$B(\alpha \wedge \alpha) = \sum_j \langle v_j'', v_j''\rangle \operatorname{Im}(\alpha_j \wedge \bar{\alpha}_j).$$

Hence we obtain:

$$((\alpha, \alpha)) = (B(\alpha \wedge \alpha), \omega) = \sum_j \langle v_j'', v_j''\rangle (\operatorname{Im}(\alpha_j \wedge \bar{\alpha}_j), \omega) = -\sum_j \langle v_j'', v_j''\rangle \|\alpha_j\|^2.$$

The other statements can be proved in a similar fashion.

4. Group cohomology, harmonic forms and the Matsushima-Murakami theorem

In the section we show that the results of the last section imply that the restrictions of Q to the non-zero Hodge pieces of $H^1(\Gamma, V)$ are given by the integral of an everywhere positive function.

We first describe the decomposition of $H^1(\Gamma, V)$ using Hodge theory. If E is a vector bundle over M we let $\mathscr{A}^r(M, E)$ denote the E-valued r-forms on M. An element of $\mathscr{A}^r(M, E)$ assigns an element of the fiber of E over x to a p-tuple of tangent vectors at $x \in M$. If E has a connection we may construct an exterior differential d on $\mathscr{A}^*(M, E)$. However $d^2 = 0$ if and only if the connection is flat.

Now let $\tilde{V}$ be the flat bundle over M associated to V. Since $\tilde{V}$ is flat we obtain a complex $\{\mathscr{A}^*(M, \tilde{V}), d\}$ with cohomology groups canonically isomorphic to the cohomology groups $H^*(M, \tilde{V})$ of M with local coefficients in V – see Raghunathan [12], Chap. VIII. We observe that the preceding remarks apply when V is replaced by any Γ-module W and $\tilde{V}$ is replaced by the associated flat bundle $\tilde{W}$.

We will be especially concerned with the case in which $W = \mathfrak{g}$. In this case we have the cup-product $H^1(\Gamma, \mathfrak{g}) \times H^1(\Gamma, \mathfrak{g}) \to H^2(\Gamma, \mathfrak{g})$ studied in Sect. 2. But we also have a multiplicative structure on $\mathscr{A}^r(M, \tilde{\mathfrak{g}})$. In the case $r = 1$ we define a bilinear function $[\ ,\ ]$ on $\mathscr{A}^1(M, \tilde{\mathfrak{g}})$ with values in $\mathscr{A}^2(\mathrm{M}, \tilde{\mathfrak{g}})$ by the formula for X, Y tangent vectors to M:

$$[\omega, \tau](X, Y) = [\omega(X), \tau(Y)] - [\omega(Y), \tau(X)].$$

The reader will verify that the above bilinear function is symmetric. In the next lemma π will denote the universal covering $\pi\colon \tilde{M} \to M$.

Lemma 4.1. *The above isomorphism carries the bilinear form* $[\ ,\ ]$ *on* $H^1(M, \tilde{\mathfrak{g}})$ *to the cup-product on* $H^1(\Gamma, \mathrm{g})$.

Proof. This is standard material but since it is essential in our proof we give the details. We define $\mathscr{A}^r(\tilde{M}, \mathfrak{g})$ to be the set of smooth $\mathfrak{g}$-valued r-forms ω on $\tilde{M}$ with the action:

$$\gamma\omega = \operatorname{Ad}\rho(\gamma)(\gamma^{-1})^* \omega.$$

Clearly one obtains an isomorphism from $\mathscr{A}^r(M, \tilde{\mathfrak{g}})$ to $\mathscr{A}^r(\tilde{M}, \mathfrak{g})^\Gamma$ by pulling back $\tilde{\mathfrak{g}}$ to $\tilde{M}$ and composing with parallel translation to the standard fiber.

We consider the double complex:

$$\{C^p(\Gamma, \mathscr{A}^q(\tilde{M}, \mathfrak{g}))\colon p, q \text{ non-negative integers}\}$$

of Eilenberg-MacLane p-cochains with values in $\mathscr{A}^q(\tilde{M}, \mathfrak{g})$ equipped with the obvious differentials. Then the above isomorphism may be realized be studying the two spectral sequences attached to the double complex.

The map from $H^1(M, \tilde{\mathfrak{g}})$ to $H^1(\Gamma, \mathfrak{g})$ is constructed as follows. Let ω be a closed 1-form on M with values in $\tilde{\mathfrak{g}}$. Choose f a smooth section of $\pi^* \tilde{\mathfrak{g}}$ such that $\mathrm{d}f = \pi^*\omega$. Define $c\colon \Gamma \to \mathscr{A}^0(\tilde{M}, \mathfrak{g})$ by $c(\gamma) = \operatorname{Ad}\rho(\gamma)^{-1} f \circ \gamma - f$. Then $c(\gamma)$ is constant, $\delta c = 0$ and so c gives rise to an element of $H^1(\Gamma, \mathfrak{g})$.

We now compute the image of $[\omega, \omega]$ in $H^2(\Gamma, \mathfrak{g})$. We note that for f as above $[f, \omega]$ is an element of $C^0(\Gamma, \mathscr{A}^1(\tilde{M}, \mathfrak{g}))$ satisfying $d([f, \omega]) = [\omega, \omega]$. But then $a = \delta[f, \omega]$ is an element of $C^1(\Gamma, \mathscr{A}^1(\tilde{M}, \mathfrak{g}))$ cohomologous to $[\omega, \omega]$ in the double complex. We find:

$$a(\gamma) = [c(\gamma), \omega].$$

But then $-[c, f]$ satisfies $-d(-[c, f]) = a$ and hence $-\delta[c, f]$ is an element of $C^2(\Gamma, \mathfrak{g})$ cohomologous to $[\omega, \omega]$ in the double complex. But a direct computation shows $\delta[c, f] = -[c, c]$. With this the lemma is proved.

We use the bilinear form B of the previous section to define a *symmetric* bilinear mapping from $\mathscr{A}^1(M, \tilde{V}) \times \mathscr{A}^1(M, \tilde{V})$ to $\mathscr{A}^2(M)$ also denoted B by:

$$B(\alpha, \beta)|_x = B(\alpha|_x, \beta|_x).$$

Here $x \in M$ and we take $U = T_x(M)$ in the discussion of the previous section.

We use the real valued bilinear form $((\ ,\ ))$ to define a symmetric bilinear form with values in $C^\infty(M)$ also denoted $((\ ,\ ))$ by:

$$((\alpha, \beta)) = (B(\alpha, \beta), \omega) \qquad \text{here } (\ ,\ ) \text{ is the Riemannian metric.}$$

Both B and $((\ ,\ ))$ factor through cohomology.

We compose $B: H^1(M, \tilde{V}) \times H^1(M, \tilde{V}) \to H^2(M, \mathbf{R})$ with integration over the cycle Z to get a quadratic form Q' on $H^1(M, \tilde{V})$.

Lemma 4.2. *Under the isomorphism between $H^1(\Gamma, V)$ and $H^1(M, \tilde{V})$ the quadratic forms Q and Q' coincide.*

Proof. The lemma follows immediately from Lemma 4.1.

We now redefine Q to be the quadratic form on $H^1(M, \tilde{V})$ constructed above. We see that our theorem will follow if we can prove that Q is positive definite.

We can give another formula for Q. Since ω^{n-1} is the Poincare dual of Z we have:

$$Q(\alpha) = \int_Z B(\alpha, \alpha) = \int_M B(\alpha, \alpha) \wedge \omega^{n-1}.$$

But $\omega^{n-1} = {}_*\omega$ hence we obtain

$$Q(\alpha) = \int_M B(\alpha, \alpha) \wedge {}_*\omega = \int_M (B(\alpha, \alpha), \omega) \operatorname{vol} = \int_M ((\alpha, \alpha)) \operatorname{vol}.$$

Here vol denotes the Riemannian volume element. Our main theorem is reduced to proving that we may choose a representative for a class α such that $((\alpha, \alpha))$ is a positive function. We will now see that the harmonic representative will work.

We now recall the Hodge theory of $H^1(M, \tilde{V})$. A good reference for the real Hodge theory is [12], Chap. VIII. For the Hodge decomposition see Murakami [10]. We have a Riemannian metric on M, we need a metric along the fibers for $\tilde{V}$. The standard positive definite Hermitian form on V is admissible, Borel-Wallach [1] [1], p. 47, and consequently we get an induced metric along the fibers and we may form a Laplacian Δ. We apply [12], Chap. VIII to conclude that each class in $H^1(M, \tilde{V})$ has a unique harmonic representative. We observe that the Hodge Theorem in [12] is stated for G semi-simple but the same proof works for G reductive with compact center.

Let us now consider the projective model for D, that is, D is the set of negative lines in $\mathbf{P}(V)$. Then we have the tautological line bundle E_- over D and the perpendicular bundle E_+ which assigns to a negative line L the vectors in $L^\perp$, the orthogonal complement of L in V for $\langle\, ,\rangle$. The bundles E_+ and E_- descend to M. We denote the corresponding bundles again by E_+ and E_-. We have then a decomposition as bundles over M:

$$\tilde{V} = E_+ \oplus E_-.$$

We let p_+ and p_- denote the corresponding bundle projections. We have an associated decomposition:

$$\mathscr{A}^*(M, \tilde{V}) = \mathscr{A}^*(M, E_+) \oplus \mathscr{A}^*(M, E_-).$$

Since E_+ and E_- are not flat bundles it would seem impossible to define cohomology groups with values in E_+ and E_-. At this point we review the general theory of Matsushima-Murakami [9] which will in fact allow us to make such a definition and get a decomposition of cohomology groups. The point will be that although the above decomposition is not compatible with d, it is compatible with Δ. To see this we will need more notation and some lemmas. Henceforth we let G denote $U(n, 1)$. The symbol K will denote $U(n) \times U(1)$, a maximal compact subgroup of G.

Let $\mathscr{A}^r(\Gamma\backslash G, K, \rho)$ denote the subspace of all V-valued smooth r-forms η on $\Gamma\backslash G$ satisfying:

(i) $\eta \circ R_k = \rho(k)^{-1} \eta$ for all $k \in K$.
(ii) $\iota(X)\eta = 0$ for all $X \in \mathfrak{k}$, the Lie algebra of K.

Here R_k denote the operation of right translation by $k \in K$ and ι denotes interior multiplication. The following fact is quite standard, Raghunathan [12], Matsushima-Murakami [9], but we include the proof for the sake of clarity.

Lemma 4.3. $\mathscr{A}^r(M, \tilde{V})$ is canonically isomorphic to $\mathscr{A}^r(\Gamma\backslash G, K, \rho)$.

Proof. We construct the map from $\mathscr{A}^r(M, \tilde{V})$ to $\mathscr{A}^r(\Gamma\backslash G, K, \rho)$.

Let $\omega \in \mathscr{A}^r(M, \tilde{V})$ and $\pi: \tilde{M} \to M$ be the universal cover. Then since $\tilde{V}$ pulled back to $\tilde{M}$ has a global parallelization we obtain a V-valued r-form ω' on $\tilde{M}$ satisfying:

$$\gamma^* \omega' = \rho(\gamma)\, \omega'.$$

We pull ω' back to G via the fibering $K \to G \to \tilde{M}$ and we obtain a V-valued form ω'' on G satisfying:

(i) $L_\gamma^* \omega'' = \rho(\gamma)\, \omega''$.
(ii) $\iota(X)\, \omega'' = 0$ for $X \in \mathfrak{k}$.

We now define ω''' a V-valued r-form on G by the formula:

$$\omega'''|_g = \rho(g)^{-1}\, \omega''|_g.$$

We leave the reader to verify that the map sending ω to ω'' is the required isomorphism.

Remark. In the last formula above it is essential that the representation ρ of Γ extend to a representation of G.

We may now write out ω''' in terms of the invariant parallelization of G to obtain an element η of $\Lambda^r \mathfrak{p}^* \otimes C^\infty(G) \otimes V$ satisfying:

(i) $[1 \otimes L_\gamma^* \otimes \rho(\gamma)] \cdot \eta = \eta$.
(ii) $[(\mathrm{Ad}\, k)^* \otimes R_k^* \otimes \rho(k)] \cdot \eta = \eta$.

Here $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ relative to the Killing form.

Let us denote the isomorphism $\mathscr{A}^r(M, \tilde{V}) \to \Lambda^r \mathfrak{p}^* \otimes C^\infty(G) \otimes V$ sending ω to η by Φ. We let q_+ and q_- denote the projections of V onto V_+ and V_-. It is apparent that:

(i) $\Phi \circ p_+ \circ \Phi^{-1} = 1 \otimes 1 \otimes q_+$.
(ii) $\Phi \circ p_- \circ \Phi^{-1} = 1 \otimes 1 \otimes q_-$.

The main ingredient in the work of Matsushima-Murakami is *Kuga's Formula* which expresses $\Phi \circ \Delta \circ \Phi^{-1}$ in terms of an operator coming from the center of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. We now describe this formula.

Let $\{X_i: i=1, 2, \ldots, s\}$ be an orthonormal basis for the restriction of the Killing form $A(\cdot, \cdot)$ to $\mathfrak{p}$ and $\{Y_j: j=1, 2, \ldots, r\}$ be a basis for $\mathfrak{k}$ such that $A(Y_i, Y_j) = -\delta_{ij}$. Consider the element C of $U(\mathfrak{g})$ given by:

$$C = \sum_{i=1}^{s} X_i^2 - \sum_{j=1}^{r} Y_j^2.$$

Then C is in the center of $U(\mathfrak{g})$; that is, it gives rise to a bi-invariant differential operator in $C^\infty(G)$. We may also consider $\rho(C) \in \operatorname{End} V$ defined by

$$\rho(C) = \sum_{i=1}^{s} \rho(X_i)^2 - \sum_{j=1}^{r} \rho(Y_j)^2.$$

Then $\rho(C)$ commutes with $\rho(X)$ for any $X \in \mathfrak{g}$. Here and in the line above we are using the symbol ρ to denote the induced representation of $\mathfrak{g}$.

Kuga's Formula then states:

$$\Phi \circ \Delta \circ \Phi^{-1} = -1 \otimes C \otimes 1 + 1 \otimes 1 \otimes \rho(C).$$

We are now in a position to decompose $H^*(M, \tilde{V})$. We first interpret the almost complex structure on D in terms of the Lie algebra $\mathfrak{k}$. We observe that we may identify $\mathbf{C}^n$ with $\mathfrak{p}$ by sending $(z_1, z_2, \ldots, z_n)$ to the matrix with the last column $(z_1, z_2, \ldots, z_n, 0)$ and last row $(\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_n, 0)$ and all other entries equal to zero. We let J be the diagonal matrix in $\mathfrak{k}$ with diagonal entries

$$\left(\frac{i}{n+1}, \ldots, \frac{i}{n+1}, \frac{-ni}{n+1}\right).$$

Then the action of $\operatorname{ad} J$ on $\mathfrak{p}$ coincides with multiplication by i on $\mathbf{C}^n$. We observe that $\mathfrak{p}$ is to be considered as a $2n$ dimensional real vector space with J operating by the usual $2n$ by $2n$ matrix.

Since V is a complex vector space we may decompose V into the eigenspaces of $\rho(J)$. Clearly this is just the decomposition $V = V^+ \oplus V^-$. The projections q_+ and q_- are then polynomials in $\rho(J)$. Since $J \in \mathfrak{g}$ we know $\rho(J)$ commutes with $\rho(C)$ and we obtain the following critical lemma.

Lemma 4.4. *Δ commutes with p_+ and p_-.*

Proof. The lemma follows since $1 \otimes C \otimes 1$ and $1 \otimes 1 \otimes \rho(C)$ commute with $1 \otimes 1 \otimes q_+$ and $1 \otimes 1 \otimes q_-$.

Corollary. *$\omega \in \mathscr{A}^r(M, \tilde{V})$ is harmonic if and only if $p_+ \omega$ and $p_- \omega$ are harmonic.*

In this way we obtain a decomposition:

$$H^r(M, \tilde{V}) = H^r(M, E_+) \oplus H^r(M, E_-).$$

We may decompose the harmonic r-forms still further by decomposing $\Lambda^r \mathfrak{p}^* \otimes \mathbf{C}$ into Hodge types.

Lemma 4.5. *Δ commutes with the projection of $\mathscr{A}^r(M, \tilde{V})$ on any Hodge type.*

Proof. After conjugation by Φ such a projection operator will be of the form $p \otimes 1 \otimes 1$ so the lemma follows from Kuga's Formula.

Returning to the case in hand we obtain a decomposition:

$$H^1(M, \tilde{V}) = H^{1,0}(M, E_+) \oplus H^{1,0}(M, E_-) \oplus H^{0,1}(M, E_+) \oplus H^{0,1}(M, E_-).$$

We now state one of our main results.

Theorem 4.1. (i) *The above decomposition is an orthogonal splitting for Q.*

(ii) *Q is negative definite on the first and last summands and positive definite on the other two.*

Proof. By Lemma 4.2 the form Q may be computed by integrating a function over M. By Theorem 3.1 this function is identically zero, positive or negative as required in the theorem. The result follows by integrating the pointwise statements of Theorem 3.1.

The positive definiteness of Q is then equivalent to the following theorem which follows from a general result of Matsushima-Murakami [9]. We leave to the reader the task of verifying that the results of [9] generalize to reductive groups with compact center.

Theorem 4.2. *$H^{1,0}(M, E_+) = H^{0,1}(M, E_-) = 0$ and consequently Q is positive definite on $H^1(M, \tilde{V})$.*

Proof. The eigenvalues of $\rho(J)$ are $i/n+1$ on V_+ and $-in/n+1$ on V_-. By Murakami [10], Theorem 6.1, or [9], Theorem 6.1 (see the discussion preceding the theorem), we find that the $(0, q)$ cohomology of M with values in $\tilde{V}$ occurs only in the highest weight space of $\rho(J)$ and the $(p, 0)$ cohomology occurs only in the lowest weight space of $\rho(J)$. With this the theorem is proved.

Corollary. *Let c be a non-zero element of $H^1(\Gamma, \mathbf{C}^{n+1})$. Then $[c, c] \neq 0$.*

Remark. The results of this paper go over to the case

$$G_0 = S(U(n, 1) \times U(k)) \quad \text{and} \quad G = SU(n+k, 1).$$

In this case $\mathscr{g}_1$ is isomorphic to V^k twisted by $U(k)$ acting from the right. We take for a Hermitian form on $\mathscr{g}_1$ the direct sum of $\langle\,,\rangle$ with itself k times. The results in Sects. 1 and 4 generalize in a straightforward manner to this case. However a modification of Lemma 2.4 is required since the centralizer of $SU(n, 1)$ in G_0 is no longer abelian. For x in $\mathscr{g}_0$ we let x'' denote the projection of x on the subalgebra $u(n, 1)$ of $\mathscr{g}_0$ (the projection is relative the obvious product decomposition). But (in the notation of Lemma 2.4) since ρ_t is normalized w_k'' is in the center of $u(n, 1)$ for $k \leqq m$. The proof of Lemma 2.4 then shows that $\delta w_{2m}'' = [v_m, v_m]''$. Since $Q(v_m)$ factors through $[v_m, v_m]''$ the above formula implies Q is isotropic. We leave the details to the reader.

Corresponding results seem to hold for embeddings of $U(n, 1)$, in the automorphism groups of other bounded dymmetric domains but we have not yet studied the question in detail.

References

1. Borel, A., Wallach, N.: Continuous cohomology, discrete subgroups, and representations of reductive groups. Ann. Math. Stud. **94**. Princeton University Press 1980
2. Bourbaki, N.: Element de mathematiques. Groupes et algebres de Lie. Chapitres 2 et 3. Paris: Hermann 1972
3. Faran, J.: Maps from the two-ball to the three-ball. Invent. Math. **80**, 441–475 (1982)
4. Goldman, W.M.: Characteristic classes and representations of discrete subgroups of Lie groups. Bull. Am. Math. Soc. **6**, 91–94 (1982)
5. Goldman, W.M.: Representations of fundamental groups of surfaces. In: Geometry and Topology, Proceedings, University of Maryland. Lect. Notes Math., vol. 1167, pp. 95–117. Berlin-Heidelberg-New York: Springer 1987
6. Johnson, D., Millson, J.J.: Deformation spaces associated to compact hyperbolic manifolds. In: Discrete groups in geometry and analysis. Proceedings of a conference held at Yale University in honor of G.D. Mostow, Progress in Math. Boston: Birkhäuser 1987
7. Kobayashi, S., Nomizu, K.: Foundations of differential geometry. Interscience tracts in pure and applied mathematics, Number 15, Vol. II
8. Kourouniotis, C.: Deformations of hyperbolic structures on manifolds of several dimensions, Math. Proc. Camb. Phil. Soc. **98**, 247–261 (1985)
9. Matsushima, Y., Murakami, S.: On vector bundle valued harmonic forms and automorphic forms on symmetric spaces. Ann. Math. **78** (2), 365–416 (1963)
10. Murakami, S.: Cohomology groups of vector-valued forms on symmetric spaces. Lecture notes, University of Chicago, 1966
11. Milnor, J.: Singular points of complex hypersurfaces. Ann. Math. Stud., vol. 61. Princeton University Press 1968
12. Raghunathan, M.S.: Discrete subgroups of Lie groups. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 68. Berlin-Heidelberg-New York: Springer 1972
13. Thurston, W.P.: The geometry and topology of three-manifolds. Princeton University Lecture Notes, 1978
14. Tits, J.: Free subgroups in linear groups. J. Algebra **20**, 250–270 (1972)
15. Webster, S.M.: On mapping an n-ball into an $(n+1)$-ball in complex space. Pacific J. Math. **81**, 267–272 (1979)
16. Weil, A.: On discrete subgroups of Lie groups. Ann. Math. **72** (2), 369–384 (1960)
17. Zimmer, R.: Ergodic theory and semisimple groups. Monographs in Mathematics. Boston: Birkhäuser, 1985
18. Artin, M.: On the solutions of analytic equations. Invent. Math. **5**, 277–291 (1968)

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Note added in proof

The special case of our conjecture in which ρ is discrete and faithful has been proved by K. Corlette in his PhD thesis at Harvard.