Dynamics on character varieties

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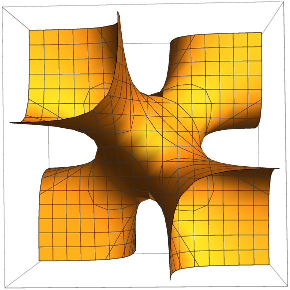
Dynamics on character varieties and geometric structures

Abstract

Classifying geometric structures on manifolds naturally leads to actions of mapping class groups on character varieties. For example complete affine structures on closed surfaces are classified by $\mathsf{GL}(2,\mathbb{Z})$-orbits on $\mathbb{R}^2$. Particularly basic are the automorphisms of the variant of the Markoff surface $x^2 + y^2 + z^2 - xyz = 20$ where the dynamics bifurcates between ergodic (level < 20) and not ergodic (level > 20).



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  - Affine structures (flat affine connections of zero torsion).
- Can introduce isolated singularities with specified cone angles — for example, translation surfaces are *very special* singular Euclidean structures.

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    - $\mathrm{Mod}(\Sigma)$-action on $\mathrm{Def}_{(G,X)}(\Sigma)$ corresponds to $\mathrm{Out}(\pi)$-action on $\mathrm{Rep}(\pi, G)$.

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- The moduli space of flat n-tori is the biquotient

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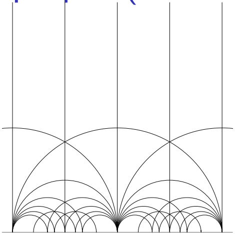

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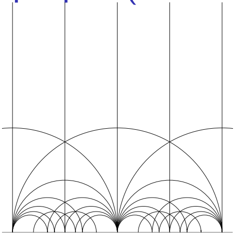

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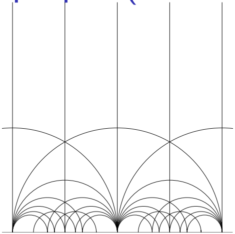

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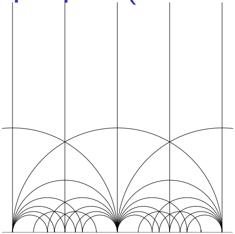

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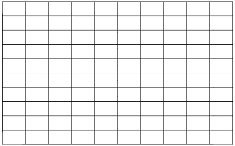
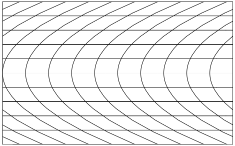
- In contrast, *complete affine* structures on with usual *linear action* of $\mathsf{GL}(2,\mathbb{Z})$. (O. Baues 2000).

Complete affine surfaces





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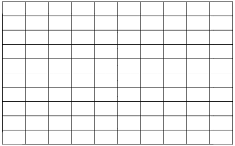 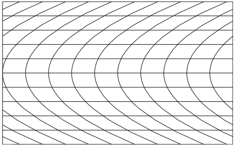

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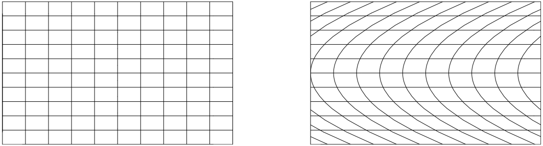

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as $T^2 \xrightarrow{\cong} \mathbb{R}^2/\phi\Lambda\phi^{-1}$. (Kuiper 1950)

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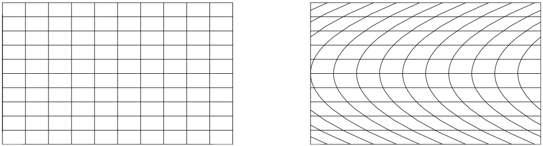

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  as $T^2 \xrightarrow{\cong} \mathbb{R}^2/\phi\Lambda\phi^{-1}$. (Kuiper 1950)
- If translation $\lambda(x, y) = (x + s, y + t)$ lies in the lattice Λ, then

$$\Big(x, y\Big) \xrightarrow{\phi\lambda\phi^{-1}} \Big(x + 2ty + (s + t^2), y + t\Big)$$

  is *affine*.

The twisted cubic cone

- Baues showed that these correspond to *invariant* affine structures on the torus *as a Lie group.*

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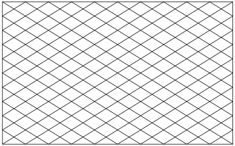
- Deligne (2021: This deformation space is naturally a *twisted cubic cone*

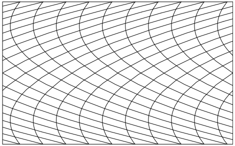
$$\left\{ [X:Y:Z:W] \in \mathbb{R}^4 \;\middle|\; XZ - Y^2 \;=\; YW - Z^2 \;=\; 0 \right\},$$

the image of the $\mathrm{GL}(2,\mathbb{Z})$-equivariant *Veronese embedding*

$$\begin{aligned} \mathbb{R}^2 &\longrightarrow \mathbb{R}^4 \\ \begin{bmatrix} x \\ y \end{bmatrix} &\longmapsto \begin{bmatrix} x^3 \\ x^2 y \\ x y^2 \\ y^3 \end{bmatrix} \end{aligned}$$

Chaotic dynamics





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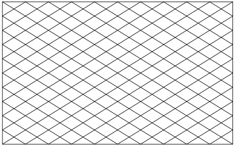

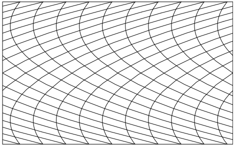

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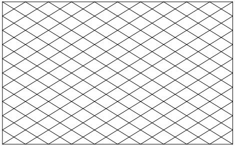

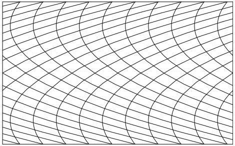

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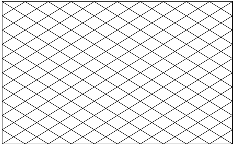

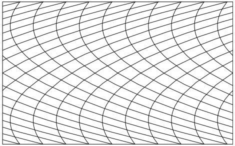

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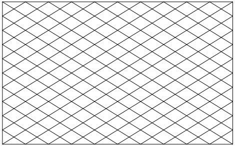

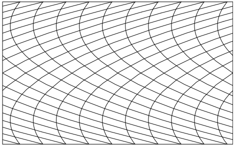

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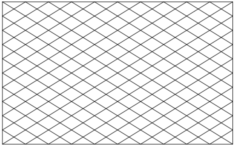

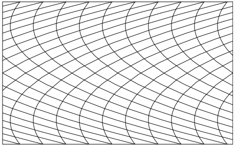

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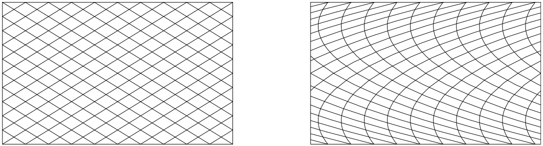

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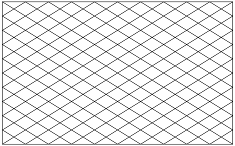

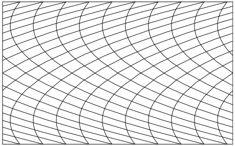

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- Therefore, the classification of geometric structures should be more insightfully regarded as a *dynamical system,* since the moduli space — its quotient — is often intractable.

Character functions and Hamiltonian twist flows

Character functions and Hamiltonian twist flows

- Elements $\gamma \in \pi_1(\Sigma)$ define *character functions* on Rep:

$$\begin{aligned} \mathsf{Rep}(\pi, G) &\xrightarrow{f_\gamma} \mathbb{R} \\ [\rho] &\mapsto \Re\big(\mathrm{Tr}\rho(\gamma)\big) \end{aligned}$$

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- For $G = \mathsf{SL}(2)$, character functions f_γ of simple γ generate coordinate ring of $\mathsf{Rep}(\pi, G)$.

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- $\mathrm{Mod}(\Sigma)$-action ergodic on regions where simple loops have elliptic holonomy.

Vogt-Fricke theorem and F_2

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- The $\mathrm{Inn}\big(SL(2)\big)$-invariant mapping

$$\begin{aligned} \mathrm{Hom}(F_2, SL(2)) &\longrightarrow \mathbb{C}^3 \\ \rho &\longmapsto \begin{bmatrix} \xi := & \mathrm{Tr}\big(\rho(X)\big) \\ \eta := & \mathrm{Tr}\big(\rho(Y)\big) \\ \zeta := & \mathrm{Tr}\big(\rho(XY)\big) \end{bmatrix} \end{aligned}$$

defines an isomorphism

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- $\mathsf{Out}(\mathsf{F}_2)$-invariant commutator trace function:

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  - Superbases are vertices in the Markoff-Bowditch tree associated to the character variety of F_2.

Invariant Poisson structure

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- The boundary trace

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determines the Poisson structure on $\mathbb{C}^3$ defined by bivector

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- $\kappa^{-1}(k)$ are the *relative character varieties.*

Vieta involutions

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- Nonlinear automorphisms of

$$\kappa(\xi, \eta, \zeta) = \xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta - 2 = k$$

generated by *involutions:*

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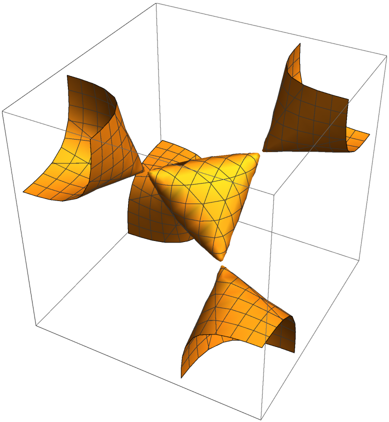
  - Fixing η and ζ yields quadratic equation in ξ;

$$\xi^2 - (\eta\zeta)\,\xi = k + 2 - \eta^2 - \zeta^2$$

whose roots ξ and $\xi' = \eta\zeta - \xi$ sum to linear coefficient $\eta\zeta$.

Cayley cubic $\xi^2 + \eta^2 + \zeta^2 - \xi\eta\zeta = 4$



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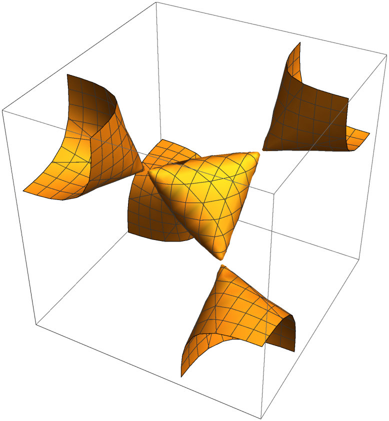

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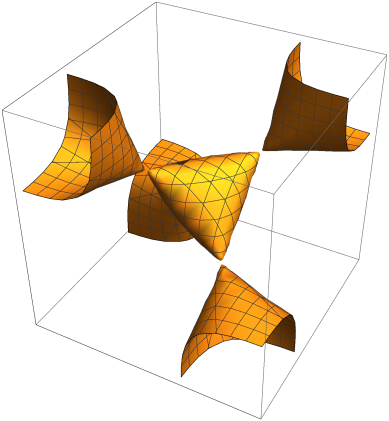

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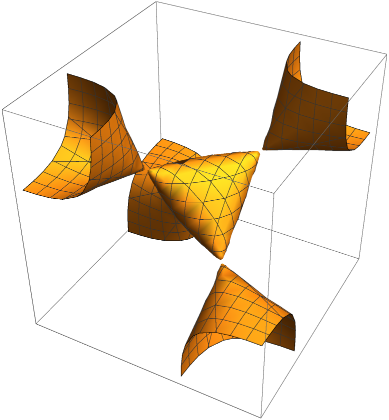
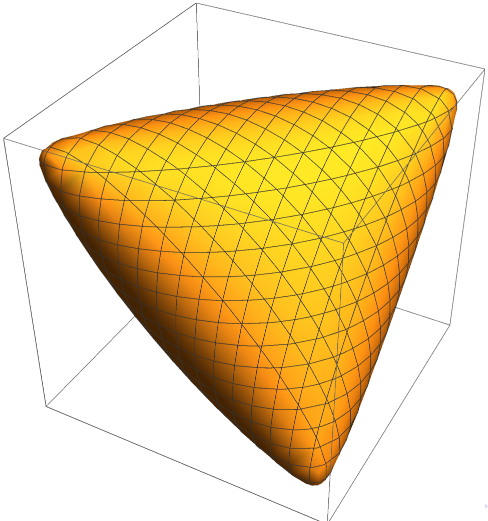
  - *Homogeneous dynamics:* $\mathrm{GL}(2, \mathbb{Z})$-action on $(\mathbb{C}^* \times \mathbb{C}^*)/(\mathbb{Z}/2)$.



$\mathbb{R}$-points: Unitary representations



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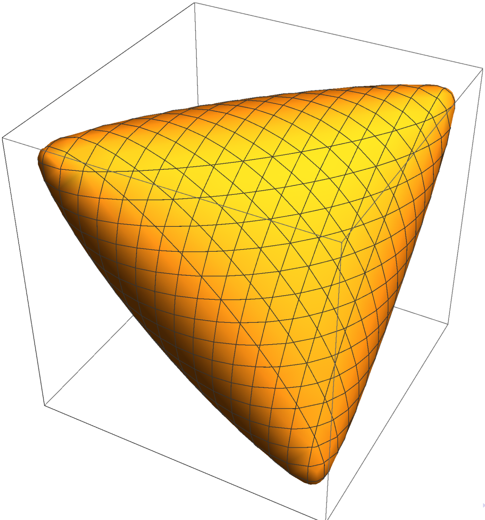

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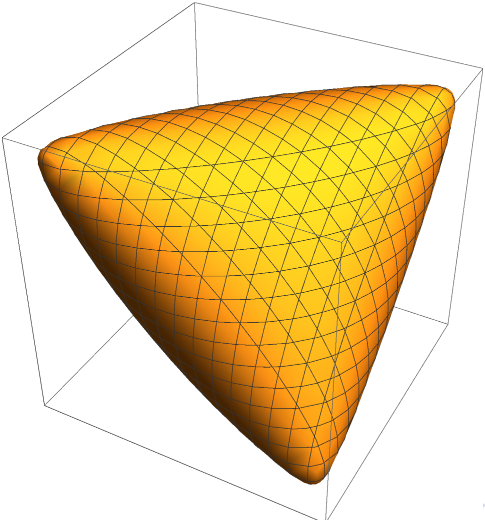
- Characters in $[-2, 2]^3$ with $\kappa \leq 2 \longleftrightarrow$ SU(2)-representations.



$\mathbb{R}$-points: Hyperbolic structures on 3-holed spheres

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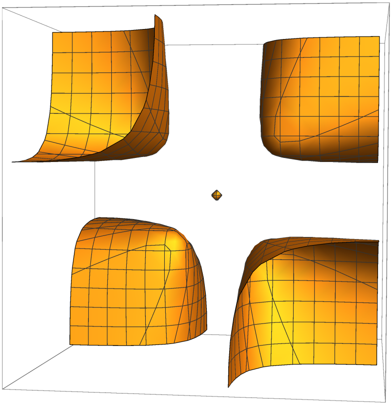
- Homotopy-equivalences $\Sigma_{1,1} \rightsquigarrow \Sigma_{0,3}$ (and other surfaces with $\pi_1 \cong F_2$) form wandering domains for $\mathrm{Out}(\mathrm{F}_2)$-action.

Example: The Markoff surface $x^2 + y^2 + z^2 = xyz$



$\mathbb{R}^3 \cap \kappa^{-1}(-2)$ parametrizes hyperbolic structures on the punctured torus. The origin $(0,0,0)$ corresponds to the unique SU(2)-representation with $k = -2$. The famous *Markoff triples* correspond to triply symmetric hyperbolic punctured tori.

Fricke orbits define wandering domains for $k > 2$

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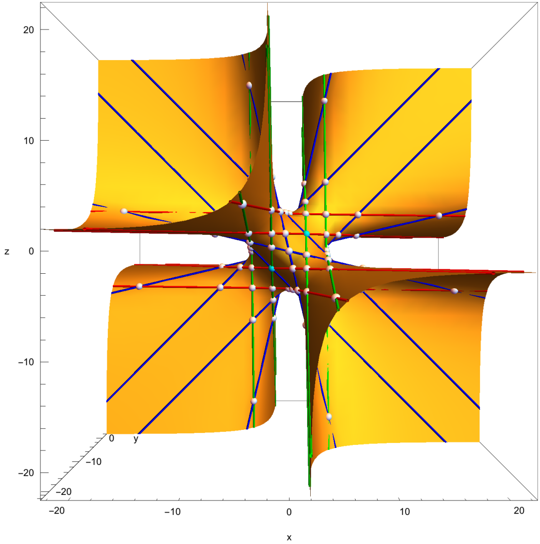
- The level surface $k = 18$ extends to the famous *Clebsch diagonal surface* in $\mathbb{C}\mathrm{P}^3$ defined by:

$$(X_0)^5+(X_1)^5+(X_2)^5+(X_3)^5+(X_4)^5 = X_0+X_1+X_2+X_3+X_4 = 0$$

in homogeneous coordinates.

$$x^2 + y^2 + z^2 - xyz = 20$$



Ergodicity for compact/noncompact groups

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- Main technique for proving ergodicity uses dynamics of Dehn twists in Mod(Σ).

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  - $\mu\big(g_t(A) \cap B\big) \to \mu(A)\mu(B)$ for A, B measurable and $g_t \to \infty$.

Happy birthday, Giovanni!