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.

A REMARKABLE FAMIY OF AFFINE CUBIC SURFACES

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Abstract. The family of affine cubics defined by

$$x^2 + y^2 + z^2 - xyz = k + 2$$

arises in several contexts, including relative $\mathsf{SL}(2,\mathbb{C})$-character varieties of the one-holed torus. We describe their geometry and symmetry.

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Introduction

The trace $\mathsf{tr}[X,Y]$ of the commutator $[X,Y] = XYX^{-1}Y^{-1}$ of two elements X, Y of $\mathsf{SL}(2,\mathbb{C})$ defines a family of affine cubic surfaces

$$\mathcal{S}_k := \{(x,y,z) \in \mathbb{C}^3 \mid \kappa(x,y,z) = k\}$$

where

$$\kappa(x,y,z) := x^2 + y^2 + z^2 - xyz - 2$$

and $x = \mathsf{tr}(X), y = \mathsf{tr}(Y), z = \mathsf{tr}(XY)$. In homogeneous coordinates X, Y, Z, W where

$$(x, y, z) \longmapsto \begin{bmatrix}\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}\end{bmatrix}$$

is the affine chart to the patch defined by $W \neq 0$: with $x = X/W, \quad y = Y/W, \quad z = Z/W$. Its projective completion $\overline{\mathcal{S}_k} \subset \mathsf{P}^3$ is defined by

$$(X^2 + Y^2 + Z^2)W - XYZ - (k+2)W^3 = 0$$

When $k \neq \pm 2$, this surface is smooth.

$\mathcal{S}_{-2}$ is the *Markoff cubic,* $\mathcal{S}_{-10/3}$ is the *Fermat cubic,* $\mathcal{S}_2$ is the *Cayley cubic,* and and $\mathcal{S}_{18}$ is the *Clebsch cubic.* When $k \geq 2$, all the lines are real and we mainly concentrate on this case. (The limiting case $\mathcal{S}_\infty$ is the *triple plane* $W^3 = 0$.)

1. Three ideal lines in the ideal plane

We begin by discussing the tritangent plane at infinity, which is *generic,* in that it three lines in general position. Thus the 27 lines divide into 3 ideal lines and 24 fninite lines. The first step in our classificaiton of lines uses these three lines to divide the finite lines into three 8-line families.

1.1. **Locus at infinity.** The structure we consider is that of the affine cubic $\mathcal{S}_k$, or, more precisely, the pair $(\overline{\mathcal{S}_k}, \mathcal{T}_\infty)$ where $\mathcal{T}_\infty$ is the plane $W = 0$ of ideal points in the fixed affine patch of $\mathbb{P}^3$. The ideal plane $\mathcal{T}_\infty$ is a *generic tritangent plane,* that is, the tangent plane to $\overline{\mathcal{S}_k}$ at three distinct points

$${}^X\mathbf{p}_\infty = \begin{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\end{bmatrix}, \qquad {}^Y\mathbf{p}_\infty = \begin{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\end{bmatrix}, \qquad {}^Z\mathbf{p}_\infty = \begin{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}\end{bmatrix}$$

$\mathcal{T}_\infty$ intersects $\overline{\mathcal{S}_k}$ in three crossing lines $XYZ = W = 0$,

$$\begin{aligned} {}^X\mathcal{I} &:= \{X = W = 0\} \\ {}^Y\mathcal{I} &:= \{Y = W = 0\} \\ {}^Z\mathcal{I} &:= \{Z = W = 0\}. \end{aligned}$$

The *ideal locus* $\overline{\mathcal{S}_k}\backslash\mathcal{S}_k$ is their union. These lines intersect in the points ${}^X\mathbf{p}_\infty, {}^Y\mathbf{p}_\infty, {}^Z\mathbf{p}_\infty$, which are the singular points of $\overline{\mathcal{S}_k}\backslash\mathcal{S}_k$. The ideal plane $\mathcal{T}_\infty$ is tangent to $\overline{\mathcal{S}_k}$ at these three points.

Although $\overline{\mathcal{S}_k}$ is smooth at ${}^X\mathbf{p}_\infty, {}^Y\mathbf{p}_\infty, {}^Z\mathbf{p}_\infty$, it does *not* intersect the ideal plane $\{W = 0\}$ transversely, and the ideal locus

$$\overline{\mathcal{S}_k}\backslash\mathcal{S}_k \;=\; \overline{\mathcal{S}_k}\cap\mathcal{T}_\infty \;=\; {}^X\mathcal{I} \;\cup {}^Y\mathcal{I} \;\cup {}^Z\mathcal{I}$$

is not a manifold.

Our context is thus the family of projective cubics $\overline{\mathcal{S}_k}$ together with a fixed generic tritangent plane; see Goldman-Toledo [3]. These pairs $(\overline{\mathcal{S}_k}, \mathcal{T}_\infty)$ also exhibit extra symmetry, which fails in general. This symmetry is reflected in the absence of linear terms in κ, which is the defining function for $\mathcal{S}_k$. Alternatively, it is reflected in the existence of ideal Eckardt points (8).

1.2. **Three families of non-ideal lines.** he basic fact we exploit is: *On each line $\ell \subset \mathcal{S}_k$, exactly one of the coordinate functions x, y, z is constant.* Thus the finite lines fall into three families labeled by X, Y, or Z. Each family consists of of eight lines, grouped into four pairs.

This elementary fact can be seen as follows. Each line ℓ on $\mathcal{S}_k$ has an ideal point

$$\ell_\infty := \overline{\ell} \cap \mathcal{T}_\infty.$$

Since the ideal locus

$$\overline{\mathcal{S}_k} \cap \mathcal{T}_\infty = {}^X\mathcal{I} \;\cup {}^Y\mathcal{I} \;\cup {}^Z\mathcal{I}\;,$$

the ideal point ℓ_∞ must lie in an ideal coordinate line. Then $\ell_\infty \in {}^X\mathcal{I}$ if and only if the affine coordinate x is constant on ℓ (and similarly for y and z).

For example, the Z-family corresponds to four planes defined by

$$\begin{aligned} z_0 &= -\sqrt{k+2} \\ z_0 &= -2 \\ z_0 &= +2 \\ z_0 &= +\sqrt{k+2}. \end{aligned} \tag{1}$$

Each plane contains two lines, and the union with the ideal line Z_∞ is a tritangent plane. The lines in the planes $z_0 = \pm\sqrt{k+2}$ we label with "$\mathcal{C}$" (for *"crossing"*). The lines in the planes $z_0 \pm 2$ we label with "$\mathcal{P}$' for (for *"parallel"*). In general the intersection of $\mathcal{S}_k$ with plane $z = z_0$ is a conic, but this conic degenerates into a union of lines at the special levels (1). Thus each special level contains two lines, either parallel ($\mathcal{P}$) or crossing ($\mathcal{C}$).

This follows easily from writing the defining equation in terms of the family of quadratic forms:

$$\mathcal{Q}_z(x,y) := x^2 - zxy + y^2 \tag{2}$$

Then $\mathcal{S}_k$ is defined by:

$$\mathcal{Q}_z(x,y) = k + 2 - z^2 \tag{3}$$

because $\kappa(x,y,z) = \mathcal{Q}_z(x,y) - z^2 - 2$. When $z = \pm 2$,

$$\begin{aligned} \mathcal{Q}_2(x,y) &= (x-y)^2 \\ \mathcal{Q}_{-2}(x,y) &= (x+y)^2. \end{aligned} \tag{4}$$

When $z = \pm\sqrt{k+2}$,

$$\begin{aligned} \mathcal{Q}_{\sqrt{k+2}}(x,y) &= (y - m^+x)(y - m^-x) \\ \mathcal{Q}_{-\sqrt{k+2}}(x,y) &= (y + m^+x)(y + m^-x) \end{aligned} \tag{5}$$

where the two *slopes,* $m^\pm \in \mathbb{Q}[\sqrt{k+2}, \sqrt{k-2}]$ are defined by:

$$m^\pm := \frac{\sqrt{k+2} \pm \sqrt{k-2}}{2} \tag{6}$$

The slopes satisfy:

$$m^+m^- = 1, \qquad m^+ + m^- = \sqrt{k+2}, \qquad m^+ - m^- = \sqrt{k-2}.$$

2. Singular points and symmetry

Next we show that $\mathcal{S}_k$ is smooth when $\neq \pm 2$. Then we discuss the group of automorphisms of κ. Autmorphisms extending to the projective completion $\overline{\mathcal{S}_k}$ form the finite group of linear autmorphisms, isomorphic to the symmetric group $\mathfrak{S}_4$.

The family $(\overline{\mathcal{S}_k}, \mathcal{S}_k)$ enjoys the symmetry of $\mathfrak{S}_3$, since the polynomial $\kappa(x,y,z)$ is symmetric. For listing the geometric objects, we exploit the 3-cycles in the cyclic *alternating group*

$$\mathfrak{A}_3 = \{(), (123), (132)\} < \mathfrak{S}_3.$$

2.1. **Critical points of κ.** The critical values of κ are ± 2. The critical points of κ are:

- The *origin* (for critical value -2)

$$\mathbf{o} := (0,0,0),$$

- and the *vertices* (for critical value +2)

$$\begin{aligned} \mathbf{c}_0 &:= (2,2,2); \\ \mathbf{c}_1 &:= (2,-2,-2); \\ \mathbf{c}_2 &:= (-2,2,-2); \\ \mathbf{c}_3 &:= (-2,-2,2). \end{aligned} \tag{7}$$

They constitute the singular sets of the Markoff cubic $\mathcal{S}_{-2}$ and the Cayley cubic $\mathcal{S}_2$ respectively. They are all nodes (ordinary double points) of the respective level sets.

The origin $\mathbf{o} = (0,0,0)$ is the character of the *quaternion representation,* given by *Pauli matrices*

$$\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

The vertices

$$\mathsf{Sing}(\mathcal{S}_{+2}) = \mathcal{C} := \{\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$$

are characters of *central* representations, that is, representations $\mathsf{F}_2 \longrightarrow \{\pm\mathbb{I}_2\}$. These representations form a group

$$\Delta \;:=\; \mathsf{Hom}\big(\mathsf{F}_2, \{\pm\mathbb{I}_2\}\big) \;\cong\; \big(\mathbb{Z}/2 \oplus \mathbb{Z}/2\big),$$

acting simply transitively on $\mathcal{C}$. In particular

$$\begin{aligned} \mathbf{c}_0 &\overset{\sigma_1}{\longleftrightarrow} \mathbf{c}_1, \qquad & \mathbf{c}_2 &\overset{\sigma_1}{\longleftrightarrow} \mathbf{c}_3 \\ \mathbf{c}_0 &\overset{\sigma_2}{\longleftrightarrow} \mathbf{c}_2, \qquad & \mathbf{c}_3 &\overset{\sigma_2}{\longleftrightarrow} \mathbf{c}_1 \\ \mathbf{c}_0 &\overset{\sigma_3}{\longleftrightarrow} \mathbf{c}_3, \qquad & \mathbf{c}_1 &\overset{\sigma_3}{\longleftrightarrow} \mathbf{c}_2. \end{aligned}$$

and thus correspond to free involutions in $\mathfrak{S}_4 = \mathsf{Aut}(\mathcal{C})$ via:

$$\begin{aligned} \sigma_1 &\longleftrightarrow (01)(23) \\ \sigma_2 &\longleftrightarrow (02)(13) \\ \sigma_3 &\longleftrightarrow (03)(12) \end{aligned}$$

As automorphisms of the family of cubics in $\mathbb{C}^3$, these are represented by *sign-change automorphisms* in $\mathsf{SL}(3,\mathbb{Z})$:

$$\sigma_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \sigma_2 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad \sigma_3 := \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Their fixed points on the cubic level sets $\mathcal{S}_k$ are the *dihedral characters*

$$
\begin{aligned}
{}^X\mathbf{d}^{\pm} &= (\pm\sqrt{k+2}, 0, 0)\\
{}^Y\mathbf{d}^{\pm} &= (0, \pm\sqrt{k+2}, 0)\\
{}^Z\mathbf{d}^{\pm} &= (0, 0, \pm\sqrt{k+2})
\end{aligned}
$$

corresponding to *dihedral representations.* For example, ${}^Z\mathbf{d}^{+} = (0, 0, \sqrt{k+2})$ corresponds to:

$$
\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \sqrt{k+2} \\ -1/\sqrt{k+2} & 0 \end{bmatrix}, \begin{bmatrix} m^- & 0 \\ 0 & m^+ \end{bmatrix}
$$

where the slopes $m^{\pm}$ are defined in (6).

3. Symmetry

We first describe the automorphisms of these cubics. The *linear automorphisms* form the symmetric group $\mathfrak{S}_4$, which extend to projective automorphisms of $\overline{\mathcal{S}_k}$. We first describe this action, which combines the triple symmetry of the three variables with the sign-change group described above. The group $\mathfrak{S}_4$ arises concretely as automorphisms of several natural 4-element sets:

- The vertices of the Cayley cubic $\mathcal{S}_2$;
- The ends of the real locus $\mathbb{R}^3 \cap \mathcal{S}_k$;
- Complementary ideal triangular regions $\overline{\mathcal{S}_k}$, that is, the components of the complement of the ideal locus in the ideal tritangent plane.

3.1. **Linear automorphisms.** These cubics all exhibit a finite symmetry group $\mathfrak{S}_4$, which can be realized as the group of *all* automorphisms of the four-element set $\mathcal{C}$, which are the vertices of a natural *tetrahedron* on the Cayley cubic $\mathcal{S}_{+2}$.

Observe first that these cubics are symmetric in the three coordinates X, Y, Z, leading to an action of the symmetric group $\mathfrak{S}_3$, which we will heavily exploit. The permutations of coordinates lead to permutations of $\mathcal{C}$ which fix $\mathbf{c}_0$ but permute $\{\mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3\}$. The rest of $\mathfrak{S}_4$ can be understood in terms of the group Δ of *sign-change automorphisms,* described above.

In terms of coordinates, the symmetric group is a split extension

$$
\Delta \vartriangleleft \mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_3,
$$

where the epimorphism $\mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_3$ is realized by permuting the three coordinates, and the kernel Δ comprises sign-changes as above. For representations, the sign-changes correspond to the action of the group of central representations $\mathsf{Hom}\big(\mathsf{F}_2, \{\pm\mathbf{1}\}\big)$ on $\mathsf{Hom}\big(\mathsf{F}_2, \mathsf{SL}(2,\mathbb{C})\big)$. The orbits comprise lifts of $\mathsf{PSL}(2,\mathbb{C})$-representations to $\mathsf{SL}(2,\mathbb{C})$ and the relative character variety $\mathcal{S}_k$ corresponds to the image of a $\Delta\cdot\mathsf{Inn}\big(\mathsf{SL}(2,\mathbb{C})\big)$-invariant subset of $\mathsf{Hom}\big(\mathsf{F}_2, \mathsf{SL}(2,\mathbb{C})\big)$ under the $\mathsf{Inn}\big(\mathsf{SL}(2,\mathbb{C})\big)$-quotient map.

The symmetric group $\mathfrak{S}_4$ is the group of *automorphisms* of the projective cubic $\overline{\mathcal{S}_k}$ for generic k, and is realized as above by *linear automorphisms* of $\mathcal{S}_k$. Both the ends of the real level sets $\overline{\mathcal{S}_k} \cap \mathbb{R}^3$...

Another finite subset invariant under the automorphism group $\mathfrak{S}_4$ is the six-element subset of *ideal Eckardt points* defined in (8). These are the ideal points of the $\mathcal{P}$-lines.

As for the six-element subset comprising dihedral characters, $\mathfrak{S}_4$ is the centralizer of an involution in $\mathfrak{S}_6$ corresponding to the six-element subset consisting of ordered pairs of distinct points of $\{0,1,2,3\}$.

3.2. **Galois automorphisms.** The lines, tritangent planes, and their intersections also enjoy *Galois symmetry* as follows. Their coordinates lie in the biquadratic field $\mathbb{Q}[\sqrt{k+2}, \sqrt{k-2}]$, at least when $\sqrt{k\pm 2} \notin \mathbb{Q}$. Its Galois group is generated by involutions

$$\sqrt{k+2} \overset{\mathcal{G}^+}{\longleftrightarrow} -\sqrt{k+2}, \qquad \sqrt{k-2} \overset{\mathcal{G}^-}{\longleftrightarrow} -\sqrt{k-2}.$$

This group, also isomorphic to $\big(\mathbb{Z}/2 \oplus \mathbb{Z}/2\big)$, acts on the configuration of lines, tritangent planes, and intersection points.

Each pair of lines in one of the three coordinate families (X, Y or Z) and four levels (1) is interchanged by the Galois involution $\mathcal{G}^-$ (see below). Observe that

$$m^\pm \overset{\mathcal{G}^-}{\longleftrightarrow} m^\mp, \qquad m^\pm \overset{\mathcal{G}^+}{\longleftrightarrow} -m^\mp$$

and $\mathcal{G}^+ \circ \mathcal{G}^- = \mathcal{G}^- \circ \mathcal{G}^+$ is an involution interchanging $m^\pm$ and $-m^\pm$.

3.3. **Vieta invoutions.** In addition to the finite groups of automorphisms which extend to projective automorphisms, the affine cubics $\mathcal{S}_k$ admit infinite groups of symmetries which define interesting dynamical systems. Namely, the coordinate projections $\mathbb{C}^3 \longrightarrow \mathbb{C}^2$ define double (branched coverings) of $\mathcal{S}_k$, and their Galois groups generate an action of the *free* 3*-generator Coxeter group* $\mathbb{Z}/2 \star \mathbb{Z}/2 \star \mathbb{Z}/2$.

Take, for example, the coordinate projection for the z-coordinate. Fix $x_0, y_0 \in \mathbb{C}$. Then the restriction of the defining cubic polynomial κ

to the coordinate line $\{(x_0,y_0)\}\times\mathbb{C}$ is quadratic. Thus the intersection

$$\mathcal{S}_k \;\cap\; \{(x_0,y_0)\}\times\mathbb{C}$$

corresponds to the pair of solutions z of the quadratic equation

$$k \;=\; \kappa(x_0,y_0,z) \;=\; z^2 - (x_0y_0)\,z \;+\; (x^2+y^2-2)$$

If z, z' are the two solutions, then

$$z + z' = x_0y_0,$$

so

$$z' = x_0y_0 - z.$$

The deck transformation of the double covering $\mathcal{S}_k \longrightarrow \mathbb{C}^2$ is the *Vieta involution.* The three Vieta involutions are:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xmapsto{{}^X\nu} \begin{bmatrix} x' := yz - x \\ y \\ z \end{bmatrix}, \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \xmapsto{{}^Y\nu} \begin{bmatrix} x \\ y' := zx - y \\ z \end{bmatrix},$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \xmapsto{{}^Z\nu} \begin{bmatrix} x \\ y \\ z' := xy - z \end{bmatrix}.$$

They generate a free Coxeter group $\mathbb{Z}/2\star\mathbb{Z}/2\star\mathbb{Z}/2.$, naturally isomorphic to the level 2 congruence subgroup $\mathsf{PGL}(2,\mathbb{Z})_{(2)}$. Specifically, the respective Vieta involutions ${}^Z\nu\;, {}^X\nu\;, {}^Y\nu\;$ are realized by the automorphisms of $\mathbb{F}_2$ and the corresponding elements of $\mathsf{PGL}(2,\mathbb{Z})$ acting on points z in the upper half-plane, respectively:

$$\left[\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \longmapsto \begin{pmatrix} X \\ Y^{-1} \\ YX^{-1} \end{pmatrix}\right] \longleftrightarrow \pm\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} : \Big(z \longmapsto -\bar{z}\Big)$$

$$\left[\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \longmapsto \begin{pmatrix} Y^{-1}X^{-1}Y^{-1} \\ Y \\ Z^{-1} \end{pmatrix}\right] \longleftrightarrow \pm\begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix} : \left(z \longmapsto \frac{\bar{z}}{1-2\bar{z}}\right)$$

$$\left[\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \longmapsto \begin{pmatrix} X^{-1} \\ X^2Y^{-1} \\ Z \end{pmatrix}\right] \longleftrightarrow \pm\begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix} : \Big(z \longmapsto 2 - \bar{z}\Big).$$

The realizations in $\mathsf{PGL}(2,\mathbb{Z})$ are reflections in the imaginary axis $i\mathbb{R}_+$, the unit semicircle $|z| = 1$, and the vertical line $1 + i\mathbb{R}_+$, respectively. (Compare [4].)

4. Two types of non-ideal lines

We classify the 24 non-ideal lines into 12 $\mathcal{P}$-lines and 12 $\mathcal{C}$-lines. Furthermore, as in §1.2, these fall into three families, corresponding to the coordinates X, Y, Z. Each family contains 4 $\mathcal{P}$-lines and 4 $\mathcal{C}$-lines.

4.1. $\mathcal{P}$-lines.

The $\mathcal{P}$-lines arise when these lines are parallel, namely when $x_0 = \pm 2$, $y_0 = \pm 2$, or $z_0 = \pm 2$ respectively. The four $\mathcal{P}$-lines in the Z-family are:

$$\begin{aligned}
{}^Z\mathcal{P}^+_+ &: \qquad z = +2 \qquad y = x + \sqrt{k-2}\\
{}^Z\mathcal{P}^+_- &: \qquad z = +2 \qquad y = x - \sqrt{k-2}\\
{}^Z\mathcal{P}^-_+ &: \qquad z = -2 \qquad y = -x + \sqrt{k-2}\\
{}^Z\mathcal{P}^-_- &: \qquad z = -2 \qquad y = -x - \sqrt{k-2}
\end{aligned}$$

arising from the factorizations (4). Apply 3-cycles in $\mathfrak{A}_3$ to obtain similar formulas for the X-family and the Y-family of $\mathcal{P}$-lines.

These lines fall into six pairs of *parallelism classes,* namely ${}^X\mathcal{P}^+_\pm$, ${}^Y\mathcal{P}^+_\pm$, and ${}^Z\mathcal{P}^+_\pm$ respectively. They meet in six *ideal Eckardt points:*

$$ {}^X\mathbf{e}^+ := {}^X\mathcal{P}^+_+ \cap {}^X\mathcal{P}^+_- = \left[\!\!\left[\begin{matrix}0\\1\\1\\0\end{matrix}\right]\!\!\right]. \qquad {}^X\mathbf{e}^- := {}^X\mathcal{P}^-_+ \cap {}^X\mathcal{P}^-_-. = \left[\!\!\left[\begin{matrix}0\\1\\-1\\0\end{matrix}\right]\!\!\right]. $$

$$ {}^Y\mathbf{e}^+ := {}^Y\mathcal{P}^+_+ \cap {}^Y\mathcal{P}^+_- = \left[\!\!\left[\begin{matrix}1\\0\\1\\0\end{matrix}\right]\!\!\right]. \qquad {}^Y\mathbf{e}^- := {}^Y\mathcal{P}^-_+ \cap {}^Y\mathcal{P}^-_- = \left[\!\!\left[\begin{matrix}1\\0\\-1\\0\end{matrix}\right]\!\!\right]. $$

(8)

$$ {}^Z\mathbf{e}^+ := {}^Z\mathcal{P}^+_+ \cap {}^Z\mathcal{P}^+_- = \left[\!\!\left[\begin{matrix}1\\1\\0\\0\end{matrix}\right]\!\!\right]. \qquad {}^Z\mathbf{e}^- := {}^Z\mathcal{P}^-_+ \cap {}^Z\mathcal{P}^-_- = \left[\!\!\left[\begin{matrix}1\\-1\\0\\0\end{matrix}\right]\!\!\right]. $$

4.2. **$\mathcal{C}$-lines.** $\mathcal{C}$-lines arise when the plane section is a degenerate conic, consisting of a a pair of *crossing* lines. Consider the $\mathcal{C}$-lines in the Z-family. Rewrite the defining equation for $\mathcal{S}_k$ as (3) using the family of quadratic forms $\mathcal{Q}_z$, defined in (2). Thus the $\mathcal{C}$-lines occur at the levels $z = z_0$ where $\mathcal{Q}_{z_0}(x,y) = 0$, that is, when $z^2 = k+2$. On these levels

$$ z_0 = +\sqrt{k+2}, $$

the factorization (5) implies that the plane section is the union of lines

$$ y = m^\pm x $$

where the slopes $m^\pm$ are defined in (6). Similarly on the level $z_0 = -\sqrt{k+2}$, the conic defined by $\mathcal{Q}_{z_0}$ degenerates into the pair of lines

$$ y = -m^\pm x $$

Thus the four $\mathcal{C}$-lines in the Z-family are:

$$ \begin{aligned} {}^Z\mathcal{C}^+_+ &: \quad z = +\sqrt{k+2} \quad && y = m^+ x \\ {}^Z\mathcal{C}^+_- &: \quad z = +\sqrt{k+2} \quad && y = m^- x \\ {}^Z\mathcal{C}^-_+ &: \quad z = -\sqrt{k+2} \quad && y = -m^+ x \\ {}^Z\mathcal{C}^-_- &: \quad z = -\sqrt{k+2} \quad && y = -m^- x \end{aligned} $$

As with $\mathcal{P}$-lines, these lines naturally pair; however the $\mathcal{C}$-lines *cross* rather than being parallel: For example, the two lines ${}^Z\mathcal{C}^{+}_{+}$ and ${}^Z\mathcal{C}^{+}_{-}$ cross, meeting in the *crossing point*

$$ {}^Z\mathbf{d}^{+} = (0,0,\sqrt{k+2}), $$

which we call a *dihedral character,* since it corresponds to the "dihedral" representation with $\rho(X),\rho(Y)$ symmetries in points and $\rho(Z)$ a transvection along the geodesic joining $\mathsf{Fix}\big(\rho(X)\big)$ and $\mathsf{Fix}\big(\rho(Y)\big)$.

These lines, together with ${}^Z\mathcal{I}$, span the tritangent plane extending $z = +\sqrt{k+2}$. Together with ${}^Z\mathcal{I}$, they span the tritangent plane extending $z = -\sqrt{k+2}$.

4.3. **Linear automorphisms and lines.** We now describe the action of $\mathfrak{S}_4$ on the set of lines. Since sign-changes act identically on each ideal coordinate line, the action of $\mathfrak{S}_4$ on the ideal lines factors through the homomorphism $\mathfrak{S}_4 \to \mathfrak{S}_3 = \mathsf{Aut}\{1,2,3\}$.

A sign-change leaves invariant each line in the corresponding family of $\mathcal{C}$-lines. Then, for example the lines in the Z-family

$$ {}^Z\mathcal{C}^{+}_{\pm} = \{z = +\sqrt{k+2},\ y = m^{\pm}x\} $$
$$ {}^Z\mathcal{C}^{-}_{\pm} = \{z = -\sqrt{k+2},\ y = -m^{\pm}x\} $$

are each invariant under $(x,y,z) \mapsto (-x,-y,z)$ which corresponds to the transposition $\sigma^{(03)} \in \mathfrak{S}_4$. The transposition $\sigma^{(12)}$ interchanges

$$ {}^Z\mathcal{C}^{\pm}_{+} \leftrightarrow {}^Z\mathcal{C}^{\pm}_{-}. $$

The two remaining sign-changes (corresponding to $\sigma^{(01)(23)}$ and $\sigma^{(02)(13)}$ in $\mathfrak{S}_4$) interchange

$$ {}^Z\mathcal{C}^{-}_{\pm} \leftrightarrow {}^Z\mathcal{C}^{+}_{\pm}. $$

Here is what happens for $\mathcal{P}$-lines. Consider the Z-family. Each

$$ {}^Z\mathcal{P}^{-}_{\pm} = \{z = -2,\ x + y = \pm\sqrt{k-2}\} $$

is invariant under the transposition (12). The sign-change σ_3 interchanges these two lines; the other sign-changes interchange

$$ {}^Z\mathcal{P}^{-}_{\pm} \longleftrightarrow {}^Z\mathcal{P}^{+}_{\pm}. $$

For ${}^Z\mathcal{P}^{+}_{\pm} = \{z = +2,\ y - x = \pm\sqrt{k-2}\}$, the transposition

$$ \begin{aligned} (03) &= \sigma_3(12) = (12)\sigma_3 \\ &= \sigma_1(12)\sigma_1 = \sigma_2(12)\sigma_2 \in \mathfrak{S}_4 \end{aligned} $$

is realized by $(x,y,z) \mapsto (-y,-x,z)$ and leaves each of these two lines invariant.

4.4. **Linear involutions and lines.** Indeed, the involutions in $\mathfrak{S}_4$ distinguish the $\mathcal{P}$-lines from the $\mathcal{C}$-lines as follows. The *alternating group* $\mathfrak{A}_4 < \mathfrak{S}_4$ consists of even permutations. The nontrivial elements of the sign-change group $\Sigma < \mathfrak{A}_4$ consists of even permutations of order two, namely products of disjoint transpositions

$$\sigma^{(ij)(k4)} \leftrightarrow \sigma_k$$

where $\{i,j,k\} = \{1,2,3\}$. The odd permutations of order two are the transpositions $\sigma^{(ij)}$ where $\{i,j\}$ is a 2-element subset of $\{1,2,3\}$ and

$$\sigma^{(k4)} \leftrightarrow \sigma_k \circ \sigma^{(ij)}.$$

Proposition 4.1. *Let $\ell \subset \mathcal{S}_k$ be a (non-ideal) line. Then ℓ is invariant under an odd involution $\Longleftrightarrow$ ℓ is a $\mathcal{P}$-line, and ℓ is invariant under an even involution $\Longleftrightarrow$ ℓ is a $\mathcal{C}$-line.*

4.5. **Vieta automorphisms and lines.** The Vieta involutions do not extend to projective space (they are not even defined by homogeneous polynomials), and therefore do not act on the ideal lines. We describe their action on a sample $\mathcal{P}$-line and a sample $\mathcal{C}$-line. As $\mathfrak{S}_4$ acts transitively on the set of $\mathcal{P}$-lines (respectively $\mathcal{C}$-lines), it suffices for the discussion to consider one sample line from each type.

First consider the $\mathcal{P}$-line ${}^Z\mathcal{P}^+_+$. The action of ${}^X\nu$ on it is:

$$\begin{bmatrix} x \\ x - \sqrt{k-2} \\ 2 \end{bmatrix} \longmapsto \begin{bmatrix} x - 2\sqrt{k-2} \\ \left(x - 2\sqrt{k-2}\right) + 2\sqrt{k-2} \\ 2 \end{bmatrix}$$

and the ${}^Y\nu$ -action is:

$$\begin{bmatrix} x \\ x - \sqrt{k-2} \\ 2 \end{bmatrix} \longmapsto \begin{bmatrix} x \\ x + \sqrt{k-2} \\ 2 \end{bmatrix}$$

Since both involutions map ${}^Z\mathcal{P}^+_\pm \to {}^Z\mathcal{P}^+_\mp$, their composition ${}^X\nu \circ {}^Y\nu$ preserves each ${}^Z\mathcal{P}^+_\pm$, translating by $-\sqrt{k-2}$ on each.

The involution ${}^Z\nu$ maps ${}^Z\mathcal{P}^+_+ +$ to a parabola:

$$\begin{bmatrix} x \\ x - \sqrt{k-2} \\ 2 \end{bmatrix} \longmapsto \begin{bmatrix} x \\ x - \sqrt{k-2} \\ \left(x + \frac{\sqrt{k-2}}{2}\right)^2 - \frac{k+6}{4} \end{bmatrix}$$

Next consider the $\mathcal{C}$-line ${}^{Z}\mathcal{C}^{+}_{+}$. Both ${}^{X}\nu$ and ${}^{Y}\nu$ map ${}^{Z}\mathcal{C}^{+}_{+} \longrightarrow {}^{Z}\mathcal{C}^{+}_{-}$:

$$\begin{bmatrix} x \\ m^+x \\ +\sqrt{k+2} \end{bmatrix} \xmapsto{{}^{X}\nu} \begin{bmatrix} (m^+)^2x \\ m^+x \\ +2\sqrt{k+2} \end{bmatrix}$$

$$\begin{bmatrix} x \\ m^+x \\ +\sqrt{k+2} \end{bmatrix} \xmapsto{{}^{Y}\nu} \begin{bmatrix} x \\ m^-x \\ +\sqrt{k+2} \end{bmatrix}$$

Since both involutions map ${}^{Z}\mathcal{C}^{+}_{+} \to {}^{Z}\mathcal{C}^{+}_{-}$, their composition ${}^{X}\nu \circ {}^{Y}\nu$ preserves ${}^{Z}\mathcal{C}^{+}_{+}$, scaling by $(m^-)^2$.

The involution ${}^{Z}\nu$ maps ${}^{Z}\mathcal{C}^{+}_{+}+$ to a parabola:

$$\begin{bmatrix} x \\ m^+x \\ +\sqrt{k+2} \end{bmatrix} \longmapsto \begin{bmatrix} x \\ m^+x \\ m^+x^2 - \sqrt{k+2} \end{bmatrix}$$

4.6. **Degeneration of lines: the Markoff surface $k = -2$.** When $k \to -2$, the levels at $\pm\sqrt{k+2}$ coalesce at the level 0, and each pair ${}^{\circ}\mathcal{C}^{+}_{\pm}, {}^{\circ}\mathcal{C}^{-}_{\pm}$ converge to a single line, for example:

$$\begin{aligned}
{}^{X}\mathcal{C}_{\pm} &:= \{z = \pm iy, x = 0.\} \\
{}^{Y}\mathcal{C}_{\pm} &:= \{x = \pm iz, y = 0.\} \\
{}^{Z}\mathcal{C}_{\pm} &:= \{y = \pm ix, z = 0.\}
\end{aligned}$$

This gives 6 $\mathcal{C}$-*lines,* each counted with multiplicity 2. The remaining 12 $\mathcal{P}$-lines are:

$$\begin{aligned}
{}^{X}\mathcal{P}^{+}_{\pm} &= \{z = y \pm 2i, x = +2\} \\
{}^{X}\mathcal{P}^{-}_{\pm} &= \{z = -y \pm 2i, x = -2\} \\
{}^{Y}\mathcal{P}^{+}_{\pm} &= \{x = z \pm 2i, y = +2\} \\
{}^{Y}\mathcal{P}^{-}_{\pm} &= \{x = -z \pm 2i, y = -2\} \\
{}^{Z}\mathcal{P}^{+}_{\pm} &= \{y = x \pm 2i, z = +2\} \\
{}^{Z}\mathcal{P}^{-}_{\pm} &= \{y = -x \pm 2i, z = -2\}
\end{aligned}$$

This gives 6 double $\mathcal{C}$-lines, 12 $\mathcal{P}$-lines and 3 ideal lines, verifying the total count of 27 lines with multiplicity. The singularity at $\mathbf{o}$ is the concurrent intersection of three double lines.

4.7. **Degeneration of lines: the Cayley surface $k = +2$.** The degeneration is more severe on $\mathcal{S}_{+2}$. In that case, $\sqrt{k-2} = 0$ implies that all the $\mathcal{P}$-lines $\mathcal{P}^+_\pm$ (respectively $\mathcal{P}^-_\pm$) coalesce. Furthermore since $m^+ = m^- = 1$, the $\mathcal{C}$-lines $\mathcal{C}^+_\pm$ (respectively $\mathcal{C}^-_\pm$) coalesce. There remain 6 *quadruple lines:*

$$
\begin{aligned}
{}^X\mathcal{P}^+ = {}^X\mathcal{C}^+ &= \{z = y, x = +2\} \\
{}^X\mathcal{P}^- = {}^X\mathcal{C}^- &= \{z = -y, x = -2\} \\
{}^Y\mathcal{P}^+ = {}^Y\mathcal{C}^+ &= \{x = z, y = +2\} \\
{}^Y\mathcal{P}^- = {}^Y\mathcal{C}^- &= \{x = -z, y = -2\} \\
{}^Z\mathcal{P}^+ = {}^Z\mathcal{C}^+ &= \{y = x, z = +2\} \\
{}^Z\mathcal{P}^- = {}^Z\mathcal{C}^- &= \{y = -x, z = -2\}
\end{aligned}
$$

5. Galois automorphisms

The lines, tritangent planes, and their intersections also enjoy *Galois symmetry* as follows. Their coordinates lie in the biquadratic field $\mathbb{Q}[\sqrt{k+2}, \sqrt{k-2}]$, at least when $\sqrt{k \pm 2} \notin \mathbb{Q}$. Its Galois group is generated by involutions

$$\sqrt{k+2} \overset{\mathcal{G}^+}{\longleftrightarrow} -\sqrt{k+2}, \qquad \sqrt{k-2} \overset{\mathcal{G}^-}{\longleftrightarrow} -\sqrt{k-2}.$$

This group, also isomorphic to $\big(\mathbb{Z}/2 \oplus \mathbb{Z}/2\big)$, acts on the configuration of lines, tritangent planes, and intersection points.

Each pair of lines in one of the three coordinate families (X, Y or Z) and four levels (1) is interchanged by the Galois involution $\mathcal{G}^-$ (see below). Observe that

$$m^\pm \overset{\mathcal{G}^-}{\longleftrightarrow} m^\mp, \qquad m^\pm \overset{\mathcal{G}^+}{\longleftrightarrow} -m^\mp$$

and $\mathcal{G}^+ \circ \mathcal{G}^- = \mathcal{G}^- \circ \mathcal{G}^+$ is an involution interchanging $m^\pm$ and $-m^\pm$.

5.1. **Galois automorphisms and lines.**

6. Example of a double-six

Recall that a *Schläfli double-six* consists of two ordered sextuples of lines $(a_1, \dots, a_6)$ and $(b_1, \dots, b_6)$ such that:

- If $i \neq j$, then a_i)(a_j.
- If $i \neq j$, then b_i)(b_j.
- a_i)(b_j if and only if $i = j$.
- a_i and b_j intersect whenever $i \neq j$.

In the last case a_i and b_j lie in a tritangent plane which we denote $\mathcal{T}_{ij}$. Write p_{ij} for the point of intersection $a_i \cap b_j$; necessarily $p_{ij} \in \mathcal{T}_{ij}$. Furthermore $\mathcal{T}_{ij}$ meets $\mathcal{S}_k$ in a third line, which we denote c_{ij}.
I am grateful to Damiano Testa for supplying the proof of the following fact:

Lemma 6.1. *If $i \neq j$, then $c_{ij} = c_{ji}$.*

Proof. Since $a_i \;)(\; a_j$ and $a_j \;)(\; b_j$, it follows that $p_{ij} \notin a_i \cup b_j$. Since $\mathcal{T}_{ij}$ is a tritangent plane to $\mathcal{S}_k$, and intersects $\mathcal{S}_k$ in

$$a_i \cup b_j \cup c_{ij},$$

$p_{ij} \in c_{ij}$. In particular a_j intersects c_{ij}.

Similarly, b_i intersects c_{ij}. Since $a_j, b_i, c_{ij} \in \mathsf{Lines}(\mathcal{S}_k)$, and they mutually cross, $c_{ij} \subset \mathcal{T}_{ji}$. Since these lines are distinct, $c_{ij} = c_{ji}$. □

Although $c_{ij} = c_{ji}$, the tritangent planes are distinct: $\mathcal{T}_{ij} \neq \mathcal{T}_{ji}$.

Here is a simple example of a double-six:

$$a_1 := {}^X\mathcal{C}^+_+, \quad a_2 := {}^X\mathcal{C}^-_+, \quad a_3 := {}^Y\mathcal{C}^+_+, \quad a_4 := {}^Y\mathcal{C}^-_+, \quad a_5 := {}^Z\mathcal{C}^+_+, \quad a_6 := {}^Z\mathcal{C}^-_+$$
$$b_1 := {}^X\mathcal{C}^+_-, \quad b_2 := {}^X\mathcal{C}^-_-, \quad b_3 := {}^Y\mathcal{C}^+_-, \quad b_4 := {}^Y\mathcal{C}^-_-, \quad b_5 := {}^Z\mathcal{C}^+_-, \quad b_6 := {}^Z\mathcal{C}^-_-$$

The c_{ij} may be computed from the following table.

	${}^X\mathcal{C}^+_+$	${}^X\mathcal{C}^-_+$	${}^Y\mathcal{C}^+_+$	${}^Y\mathcal{C}^-_+$	${}^Z\mathcal{C}^+_+$	${}^Z\mathcal{C}^-_+$
${}^X\mathcal{C}^-_-$		${}^X\mathcal{I}$	${}^Z\mathcal{P}^+_-$	${}^Z\mathcal{P}^-_+$	${}^Y\mathcal{P}^+_-$	${}^Y\mathcal{P}^-_-$
${}^X\mathcal{C}^+_-$	${}^X\mathcal{I}$		${}^Z\mathcal{P}^-_-$	${}^Z\mathcal{P}^+_+$	${}^Y\mathcal{P}^-_+$	${}^Y\mathcal{P}^+_+$
${}^Y\mathcal{C}^-_-$	${}^Z\mathcal{P}^+_-$	${}^Z\mathcal{P}^-_-$		${}^Y\mathcal{I}$	${}^X\mathcal{P}^+_-$	${}^X\mathcal{P}^-_+$
${}^Y\mathcal{C}^+_-$	${}^Z\mathcal{P}^-_+$	${}^Z\mathcal{P}^+_+$	${}^Y\mathcal{I}$		${}^X\mathcal{P}^-_-$	${}^X\mathcal{P}^+_+$
${}^Z\mathcal{C}^-_-$	${}^Y\mathcal{P}^+_-$	${}^Y\mathcal{P}^-_+$	${}^X\mathcal{P}^+_-$	${}^X\mathcal{P}^-_-$		${}^Z\mathcal{I}$
${}^Z\mathcal{C}^+_-$	${}^Y\mathcal{P}^-_-$	${}^Y\mathcal{P}^+_+$	${}^X\mathcal{P}^-_+$	${}^X\mathcal{P}^+_+$	${}^Z\mathcal{I}$	

TABLE 1. The Principal Double-Six

Clebsch's *Diagonal Cubic Surface* is the interesection of the cubic hypersurface in $\mathbb{C}\mathsf{P}^3$

$$(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 \;=\; 0,$$

with the hyperplane

$$U_0 + U_1 + U_2 + U_3 + U_4 \;=\; 0.$$

It clearly enjoys $\mathfrak{S}_5$-symmetry, which is one of the maximal symmetry groups of a smooth projective cubic surface ($\#\mathfrak{S}_5 = 5! = 120$).

It has 10 Eckardt points. In addition to the 6 ideal Eckardt points, there are 4 non-ideal Eckardt points $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

We relate the homogeneous coordinates U_0, U_1, U_2, U_3, U_4 to the our original homogeneous coordinates X, Y, Z, W:

$$\begin{aligned} U_0 &\longmapsto W \\ U_1 &\longmapsto (-2W - X - Y + Z)/8 \\ U_2 &\longmapsto (-2W - X + Y - Z)/8 \\ U_3 &\longmapsto (-2W + X - Y - Z)/8 \\ U_4 &\longmapsto (-2W + X + Y + Z)/8 \end{aligned} \tag{9}$$

The transposition $U_0 \leftrightarrow U_4$ corresponds to the involution

$$(x, y, z) \xleftrightarrow{(04)} \left(\frac{x}{x+y+z-2}, \frac{y}{x+y+z-2}, \frac{z}{x+y+z-2} \right)$$

which fixes the tritangent plane $x + y + z = 6$ (equal to $\mathcal{G}^{-}\mathcal{T}_{13}$). It maps the ideal tritangent plane $\mathcal{T}_\infty$ to the hyperplane $x + y + z = 2$, the plane containing the three Eckardt points $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. The subgroup $\mathfrak{S}_4 < \mathfrak{S}_5$ consists of the linear automorphisms discussed earlier, and $\mathfrak{S}_5 = \langle \mathfrak{S}_4, (04) \rangle$.

6.1. **Dynamical significance.** The tritangent plane $\mathcal{T}_{13}$ defined by:

$$x + y + z + (2 + \sqrt{k-2}) = 0$$

containing the three lines:

$$\begin{aligned} {}^{Z}\mathcal{P}_{+}^{-} &:= \{ z = -2, \qquad x + y + \sqrt{k-2} = 0 \} \\ {}^{X}\mathcal{P}_{+}^{-} &:= \{ x = -2, \qquad y + z + \sqrt{k-2} = 0 \} \\ {}^{Y}\mathcal{P}_{+}^{-} &:= \{ y = -2, \qquad z + x + \sqrt{k-2} = 0 \} \end{aligned}$$

is dynamically interesting, for $k \geq 18$. The orthant Ω defined by $x, y, z \leq -2$ parametrizes the Fricke space of the 3-holed sphere with the *standard marking,* the one whose generators correspond to the boundary components. It meets $\mathcal{S}_k$ in a wandering domain for the action of

$$\Gamma := \mathsf{Out}(\mathsf{F}_2) \cong \pi_0\big(\mathsf{Homeo}(S)\big) \cong \mathsf{GL}(2,\mathbb{Z})$$

and is bounded by T_{13}. Geometrically, points in the orbit $\Gamma\Omega$ correspond to homotopy equivalences $S \rightsquigarrow M$, where M is a complete hyperbolic surface homeomorphic to a three-holed sphere.

The Eckardt point $\mathbf{e}_0 := (-2,-2,-2)$ in the Clebsch cubic $\mathcal{S}_{18}$ corresponds to the complete finite area 3-*punctured* sphere M. This Eckardt point arises as the domain Ω collapses as $k \searrow 18$.) The four $\mathcal{P}$-lines in each coordinate family bound an open annulus, whose levels are ellipses. The corresponding cyclic group of Dehn twists acts minimally (and ergodically) on almost every level ellipse. This leads to chaotic dynamics (ergodicity with respect to the Poisson measure arising from the invariant function κ and Euclidean volume form) on the complement of the orbit $\Gamma\cdot\Omega$ of the wandering domain.

7. The Fermat surface

This is the cubic surface in $\mathbb{C}\mathsf{P}^3$

$$(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 \;=\; 0. \tag{10}$$

The plane

$$U_0 + U_1 + U_2 + U_3 \;=\; 0, \tag{11}$$

is a generic triangent plane $\mathcal{T}$ and $\mathcal{S}_{-10/3}$ is the corresponding affine cubic surface. $\mathcal{T}$ meets the surface defined by (10) in the locus

$$(U_1+U_2)(U_2+U_3)(U_3+U_1) = 0,$$

a union of three crossing lines, proving $\mathcal{T}$ is generic, as claimed.

Idealizing (11), the resulting affine patch has affine coordinates (u_1,u_2,u_3) with chart:

$$\begin{aligned} \mathbb{C}^3 &\longrightarrow \mathbb{C}\mathsf{P}^3 \setminus \mathcal{T} \\ (u_1,u_2,u_3) &\longmapsto \left[\!\!\left[\begin{matrix} 1-u_1-u_2-u_3 \\ u_1 \\ u_2 \\ u_3 \end{matrix}\right]\!\!\right] \end{aligned}$$

so (10) becomes:

$$(1-u_1-u_2-u_3)^3 + (u_1)^3 + (u_1)^3 + (u_2)^3 + (u_2)^3 \;=\; 0.$$

Apply the substitution

$$\begin{aligned} x &:= 4(u_2 + u_3) - 2 \\ y &:= 4(u_3 + u_1) - 2 \\ x &:= 4(u_2 + u_3) - 2 \end{aligned}$$

to obtain:

$$(1 - u_1 - u_2 - u_3)^3 + (u_1)^3 + (u_1)^3 + (u_2)^3 + (u_2)^3 \;=\; \frac{3\,\kappa(x,y,z) + 10}{64}.$$

Therefore this affine piece of the Fermat surface identifies with $\mathcal{S}_{-10/3}$.

8. Sylvester pentahedral form

This can be cast in a more general context using the *Sylvester pentahedron.* Namely the substitution (9) maps the projective variety with homogeneous coordinates U_0, U_1, U_2, U_3, U_4 to the the original homogeneous coordinates X, Y, Z, W. In particular (9) maps the homogeneous equation

$$\frac{3k+10}{64}(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 \;=\; 0,$$

to the inhomogeneous equation

$$x^2 + y^2 + z^2 - xyz - 2 = k$$

where $x = X/W, y = Y/W, z = Z/W$. The *Sylvester pentahedron* is the configuration formed by the hyperplanes defined by the U_i.

In particular the case $k = -10/3$ corresponds to the case when the coefficient of the U_0-term vanishes (the Fermat surface), the case $k = -2$ (the Markoff surface) corresponds to

$$\frac{1}{8}(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 \;=\; 0,$$

the case $k = 2$ (the Cayley surface) corresponds to

$$\frac{1}{2}(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 \;=\; 0,$$

and the Clebsch surface ($k = 18$) corresponds to

$$(U_0)^3 + (U_1)^3 + (U_2)^3 + (U_3)^3 + (U_4)^3 \;=\; 0,$$

the surface with full $\mathfrak{S}_5$-symmetry.

Notation

Ideal lines: ${}^X\mathcal{I}$, ${}^Y\mathcal{I}$, ${}^Z\mathcal{I}$
Coordinate ideal points: ${}^X\mathbf{p}_\infty, {}^Y\mathbf{p}_\infty, {}^Z\mathbf{p}_\infty$
$\mathcal{C}$-lines: ${}^X\mathcal{C}^\pm_\pm$, ${}^Y\mathcal{C}^\pm_\pm$, ${}^Z\mathcal{C}^\pm_\pm$
$\mathcal{P}$-lines: ${}^X\mathcal{P}^\pm_\pm$, ${}^Y\mathcal{P}^\pm_\pm$, ${}^Z\mathcal{P}^\pm_\pm$
Ideal tritangent plane: $\mathfrak{T}_\infty$
Ideal Eckardt points: ${}^X\mathbf{e}^\pm, {}^Y\mathbf{e}^\pm, {}^Z\mathbf{e}^\pm$
Finite Eckardt points: $\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
Symmetric groups: $\mathfrak{S}_3, \mathfrak{S}_4, \mathfrak{S}_5$
Critical points of κ: $\mathbf{c}_0, \mathbf{c}_1, \mathbf{c}_2, \mathbf{c}_3$
Dihedral characters: ${}^X\mathbf{d}^\pm, {}^Y\mathbf{d}^\pm, {}^Z\mathbf{d}^\pm$
Sign-changes $\sigma_1, \sigma_2, \sigma_3$ comprise Δ realizing $\big(\mathbb{Z}/2 \oplus \mathbb{Z}/2\big) \triangleleft \mathfrak{S}_4$
Vieta involutions ${}^X\nu$, ${}^Y\nu$, ${}^Z\nu$

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