

Stretching Three-Holed Spheres and the Margulis Invariant

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ABSTRACT. This paper applies the authors' forthcoming work, "Affine deformations of a three-holed sphere" in Lorentzian geometry to prove a result in hyperbolic geometry. Namely, an infinitesimal deformation of a hyperbolic structure of a three-holed sphere which infinitesimally lengthens the three boundary components infinitesimally lengthens every closed geodesic. The proof interprets the derivative of the geodesic length function as the Margulis invariant (signed marked Lorentzian length spectrum) of the corresponding affine deformation. The aforementioned results imply that the affine deformation is proper, and hence by Margulis's Opposite Sign Lemma, every closed geodesic infinitesimally lengthens.

1. Introduction

This note concerns an application of the emerging theory of *complete flat Lorentz 3-manifolds* to hyperbolic geometry on surfaces.

We shall apply our forthcoming paper [6] to prove the following simple result:

THEOREM 1.1. *Let Σ be a three-holed sphere. Consider a one-parameter family Σ_t of marked hyperbolic structures on Σ . For each $\gamma \in \pi_1(\Sigma)$ denote the length of the closed geodesic corresponding to γ by $\ell(\gamma)$.*

Suppose that for each ∂_i corresponding to a component of $\partial\Sigma$,

$$\left. \frac{d\ell(\partial_i)}{dt} \right|_{t=0} > 0.$$

Then for every essential closed curve γ ,

$$\left. \frac{d\ell(\gamma)}{dt} \right|_{t=0} > 0.$$

A *complete flat Lorentz 3-manifold* is a geodesically complete Lorentzian 3-manifold of zero curvature. Such a manifold is a quotient $M = \mathbb{M}^{2,1}/\Gamma$ of $(2 +$

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1)-dimensional Minkowski space $\mathbb{M}^{2,1}$ by a discrete group Γ of isometries acting properly and freely on $\mathbb{M}^{2,1}$. Recall that $(2+1)$ -dimensional Minkowski space is a geodesically complete simply connected Lorentzian manifold of zero curvature.

Alternatively, $\mathbb{M}^{2,1}$ is a 3-dimensional affine space, together with a quadratic form of signature $(2, 1)$ on the vector space of translations. We shall call such an inner product space a *Lorentzian vector space*. Every such Lorentzian vector space is isomorphic to \mathbb{R}^3 with inner product

$$\mathbb{B}(x, y) = x_1y_1 + x_2y_2 - x_3y_3,$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

We denote this Lorentzian vector space by $\mathbb{R}^{2,1}$. The group of automorphisms of $\mathbb{R}^{2,1}$ is the orthogonal group $O(2, 1)$.

The deformation theory of hyperbolic structures on surfaces intimately relates to discrete groups of isometries of $\mathbb{M}^{2,1}$. Every quotient of $\mathbb{M}^{2,1}$ by a discrete group Γ of isometries acting properly determines a complete noncompact hyperbolic surface Σ . In [6], quotients are classified when the corresponding hyperbolic surface Σ is homeomorphic to a 3-holed sphere. Here we describe this classification in terms of deformations of hyperbolic structures on the three-holed sphere.

The interplay between the two deformation theories owes to the identification (first exploited in this context in [18] and [15]) of the Lie algebra of Killing vector fields on the hyperbolic plane \mathbb{H}^2 as a Lorentzian vector space. Infinitesimal deformations of a hyperbolic surface Σ correspond to *affine deformations* of the holonomy representation $\pi_1(\Sigma) \xrightarrow{\rho} \text{Isom}(\mathbb{H}^2)$. By [14], complete flat Lorentz 3-manifolds M fall into two distinct types. The first type arises when $\pi_1(M)$ is solvable, in which the classification is a simple exercise in linear algebra. The second type, which we call *nonelementary*, arises from an affine deformation ρ of a homomorphism $\pi_1(M) \xrightarrow{\rho_0} \text{Isom}(\mathbb{H}^2) \cong \text{SO}(2, 1)^0$ satisfying the following conditions:

- ρ_0 is injective;
- The image Γ_0 of ρ_0 is a discrete subgroup of $\text{SO}(2, 1)^0$;
- ρ defines a *proper action* of $\pi_1(M)$ on $\mathbb{M}^{2,1}$.

By Mess [22], the hyperbolic surface $\Sigma := \mathbb{H}^2/\Gamma_0$ is noncompact. Thus isometry classes of nonelementary complete flat Lorentz 3-manifolds M correspond to conjugacy classes of affine deformations ρ of discrete embeddings of $\pi_1(M)$ in $\text{Isom}(\mathbb{H}^2) \cong \text{SO}(2, 1)^0$, defining *proper actions* of $\pi_1(M)$ on $\mathbb{M}^{2,1}$. Henceforth we refer to such affine deformations ρ as *proper deformations*.

Determining which affine deformations are proper is a fundamental and difficult problem. When $\pi_1(\Sigma) \xrightarrow{\rho_0} \text{Isom}(\mathbb{H}^2)$ embeds $\pi_1(\Sigma)$ in a Schottky group, [17] provides criteria in terms of an extension of an invariant discovered by Margulis [20, 21]. One of the main results of [6] is that these criteria take a particularly simple form when Σ is a 3-holed sphere.

Nonelementary flat Lorentz 3-manifolds behave like hyperbolic surfaces in many ways. For example, if $\gamma \in \pi_1(M)$ does not correspond to a cusp of Σ , then γ corresponds to a closed geodesic in Σ , and we denote its length by $\ell(\gamma)$. In M , γ corresponds to a closed geodesic with respect to the induced flat Lorentz metric on M . This geodesic is spacelike and has a well-defined positive length.

Margulis defined, for *any* affine deformation ρ , a class function

$$\{\gamma \in \pi_1(M) \mid \rho_0(\gamma) \text{ is hyperbolic}\} \xrightarrow{\alpha_\rho} \mathbb{R}$$

with many remarkable properties.

For example, $\alpha(\gamma) \neq 0$ if and only if the cyclic group $\langle \rho(\gamma) \rangle$ acts freely on $\mathbb{M}^{2,1}$. When ρ is proper, then $|\alpha(\gamma)|$ is the length of the closed geodesic in M corresponding to γ . Furthermore in this case, either the values of α are all positive or all negative.

Affine deformations of the holonomy representation $\pi_1(\Sigma) \xrightarrow{\rho_0} \text{Isom}(\mathbb{H}^2)$ of a complete hyperbolic surface Σ correspond to infinitesimal deformations of the hyperbolic manifold Σ as follows. As first observed by Weil [25, 26] (see also Raghunathan [23]), infinitesimal deformations of the geometric structure on Σ correspond to vectors in the Zariski tangent space to

$$\text{Hom}(\pi_1(\Sigma), \text{Isom}(\mathbb{H}^2))$$

at ρ_0 . Weil computed this tangent space as the cohomology

$$H^1(\Sigma, \mathfrak{g})$$

with coefficients in the local system corresponding to the Lie algebra $\mathfrak{g} \cong \mathfrak{sl}(2, \mathbb{R})$ of Killing vector fields, that is, *infinitesimal isometries* of \mathbb{H}^2 . This local system identifies with the $\pi_1(M) \cong \pi_1(\Sigma)$ -module corresponding to the Lorentzian vector space $\mathbb{R}^{2,1}$ with the action given by ρ_0 .

For example suppose that $u \in Z^1(\pi_1(\Sigma), \mathbb{R}^{2,1})$ is a cocycle, the corresponding affine deformation is explicitly given as:

$$\rho(\gamma) : x \longmapsto \gamma(x) + u(\gamma).$$

Suppose that ρ_t is a smooth path in $\text{Hom}(\pi_1(\Sigma), \text{Isom}(\mathbb{H}^2))$, of holonomies of hyperbolic structures, with geodesic length functions

$$\pi_1(\Sigma) \xrightarrow{\ell_t} \mathbb{R}$$

and velocity vector u .

In this interpretation, Margulis's invariant $\alpha(\gamma)$ is the rate of change of the geodesic length function $\ell_t(\gamma)$ under the smooth path ρ_t above:

$$\alpha(\gamma) = \left. \frac{d\ell_t(\gamma)}{dt} \right|_{t=0}.$$

Thus, if $\alpha(\gamma) > 0$ for $\gamma \neq 1$, then the length of γ increases to first order under the deformation of hyperbolic structures corresponding to ρ_t .

Let Σ be a noncompact surface with a complete hyperbolic metric. If there exists a simple closed geodesic γ bounding a noncompact part of Σ homeomorphic to a cylinder, that part of the surface is called an *end* with corresponding geodesic γ . If there is a simple closed curve bounding a noncompact part of Σ homeomorphic to a cylinder but with finite area, that part of the surface is called a *cusps*. Ends have hyperbolic holonomy and cusps, parabolic holonomy.

We can naturally associate a length to each end of Σ , namely, the length of its corresponding closed geodesic. Extend this notion of length to a cusp by declaring a cusp to be of length zero. So given a path of holonomy representations, we can say that the cusp *lengthens* if its image along the path deforms to an end.

In this language, we may restate Theorem 1.1 as follows.

THEOREM 1.2. *Let Σ be a complete surface homeomorphic to a three-holed sphere with hyperbolic structure induced by the holonomy representation*

$$\rho : \pi_1(\Sigma) \longrightarrow \text{Isom}(\mathbb{H}^2).$$

Suppose ρ_t is a path of holonomy representations such that $\rho_0 = \rho$.

If the lengths of the three components $\partial_1, \partial_2, \partial_3$ of $\partial\Sigma$ are increasing along ρ_t , then up to first order, the length of every closed geodesic is increasing.

Using an argument inspired by Thurston [24], we will in fact prove the following extension of Theorem 1.2.

THEOREM 1.3. *Let Σ be a complete surface homeomorphic to a three-holed sphere with boundary components $\partial_1, \partial_2, \partial_3$. Let ρ_t be a path of holonomy representations. If the lengths of the ∂_i are increased (respectively not decreased), then every closed geodesic on Σ is increased (respectively not decreased).*

2. Background

2.1. The geometry of $\mathbb{R}^{2,1}$. Let $\mathbb{M}^{2,1}$ denote Minkowski (2+1)-space, the three-dimensional affine space with the following additional structure. Its associated vector space of directions is

$$\mathbb{R}^{2,1} = \{p - q \mid p, q \in \mathbb{M}^{2,1}\}.$$

This vector space is isomorphic to \mathbb{R}^3 as a vector space with the standard Lorentzian inner product:

$$\mathbb{B}(x, y) = x_1y_1 + x_2y_2 - x_3y_3,$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

A non-zero vector x is said to be *null* (respectively *timelike*, *spacelike*) if $\mathbb{B}(x, x) = 0$ (respectively $\mathbb{B}(x, x) < 0$, $\mathbb{B}(x, x) > 0$).

2.2. Lorentzian transformations and affine deformations. Let $\text{Aff}(\mathbb{M}^{2,1})$ denote the group of all affine transformations that preserve the Lorentzian inner product on the space of directions; $\text{Aff}(\mathbb{M}^{2,1})$ is isomorphic to $\text{O}(2, 1) \ltimes \mathbb{R}^{2,1}$. We shall restrict our attention to those transformations whose linear parts are in $\text{SO}(2, 1)^0$, thus preserving orientation and time-orientation.

Denote projection onto the *linear part* of an affine transformation by:

$$\text{Aff}(\mathbb{M}^{2,1}) \xrightarrow{\mathbb{L}} \text{O}(2, 1).$$

Recall that the upper sheet of the hyperboloid of unit-timelike vectors in $\mathbb{R}^{2,1}$ is a model for the hyperbolic plane \mathbb{H}^2 . The resulting isomorphism between $\text{SO}(2, 1)^0$ and $\text{Isom}(\mathbb{H}^2)$ gives rise to the following terminology. (Consult [18], for example, for an explicit isomorphism.)

DEFINITION 2.1. Let $g \in \text{SO}(2, 1)^0$ be a nonidentity element;

- g is *hyperbolic* if it has three, distinct real eigenvalues;
- g is *parabolic* if its only eigenvalue is 1;
- g is *elliptic* otherwise.

We also call $\gamma \in \text{Aff}(\mathbb{M}^{2,1})$ *hyperbolic* (respectively *parabolic*, *elliptic*) if its linear part $\mathbb{L}(\gamma)$ is hyperbolic (respectively parabolic, elliptic).

Let $\Gamma_0 \subset \mathrm{O}(2, 1)$ be a subgroup. An *affine deformation* of Γ_0 is a representation

$$\rho : \Gamma_0 \longrightarrow \mathrm{Aff}(\mathbb{M}^{2,1}).$$

For $g \in \Gamma_0$, write

$$\rho(g)(x) = g(x) + u(g)$$

where $u(g) \in \mathbb{R}^{2,1}$. Then u is a cocycle of Γ_0 with coefficients in the Γ_0 -module $\mathbb{R}^{2,1}$ corresponding to the linear action of Γ_0 . In this way affine deformations of Γ_0 correspond to cocycles in $Z^1(\Gamma_0, \mathbb{R}^{2,1})$ and translational conjugacy classes of affine deformations correspond to cohomology classes in $H^1(\Gamma_0, \mathbb{R}^{2,1})$.

By extension, if $\Gamma_0 = \rho_0(\pi_1(\Sigma))$, we will call ρ an affine deformation of the holonomy representation ρ_0 .

2.3. The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ as $\mathbb{R}^{2,1}$. The Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ is the tangent space to $\mathrm{PSL}(2, \mathbb{R})$ at the identity and consists of the set of traceless 2×2 matrices. The three-dimensional vector space has a natural inner product, the Killing form, defined to be

$$(2.1) \quad \langle V, W \rangle = \frac{1}{2} \mathrm{Tr}(V \cdot W).$$

A basis for $\mathfrak{sl}(2, \mathbb{R})$ is given by

$$(2.2) \quad E_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Evidently, $\langle E_1, E_1 \rangle = \langle E_2, E_2 \rangle = 1$, $\langle E_3, E_3 \rangle = -1$ and $\langle E_i, E_j \rangle = 0$ for $i \neq j$. That is, $\mathfrak{sl}(2, \mathbb{R})$ is isomorphic to $\mathbb{R}^{2,1}$ as a vector space

$$\left\{ v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right\} \leftrightarrow \{ xE_1 + yE_2 + zE_3 = V \}.$$

The adjoint action of $\mathrm{PSL}(2, \mathbb{R})$ on $\mathfrak{sl}(2, \mathbb{R})$:

$$g(V) = gVg^{-1}$$

corresponds to the linear action of $\mathrm{SO}(2, 1)^0$ on $\mathbb{R}^{2,1}$.

Using these identifications, set:

$$\begin{aligned} \mathbf{G} &\cong \mathrm{PSL}(2, \mathbb{R}) \cong \mathrm{SO}(2, 1)^0 \\ \mathfrak{g} &\cong \mathfrak{sl}(2, \mathbb{R}) \cong \mathbb{R}^{2,1}. \end{aligned}$$

2.4. The Margulis invariant. The Margulis invariant is a measure of an affine transformation's signed Lorentzian displacement in $\mathbb{M}^{2,1}$. Originally defined by Margulis for hyperbolic transformations [20, 21], it admits an extension to parabolic transformations [4].

Let $g \in \mathbf{G}$ be a non-elliptic element. Lift g to a representative in $\mathrm{SL}(2, \mathbb{R})$; then the following element of \mathfrak{g} is a g -invariant vector which is independent of choice of lift:

$$F_g = \sigma(g) \left(g - \frac{\mathrm{Tr}(g)}{2} I \right)$$

where $\sigma(g)$ is the sign of the trace of the lift.

Now let $\Gamma_0 \subset \mathbf{G}$ such that every element other than the identity is non-elliptic. Let ρ be an affine deformation of Γ_0 , with corresponding $u \in Z^1(\Gamma_0, \mathbb{R}^{2,1})$. Given

the above identification $\mathfrak{g} \cong \mathbb{R}^{2,1}$, we may also write $u \in Z^1(\Gamma_0, \mathfrak{g})$. We define the *non-normalized Margulis invariant* of $\rho(g) \in \rho(\Gamma_0)$ to be:

$$(2.3) \quad \tilde{\alpha}_\rho(g) = \langle u(g), F_g \rangle.$$

(In [4], the non-normalized invariant is a functional on a fixed line, rather than a value.)

If $\rho(g)$ happens to be hyperbolic, then F_g is spacelike and we may replace it by the unit-spacelike vector:

$$X_g^0 = \frac{2\sigma(g)}{\sqrt{\text{Tr}(g)^2 - 4}} \left(g - \frac{\text{Tr}(g)}{2} I \right)$$

obtaining the *normalized Margulis invariant*:

$$(2.4) \quad \alpha_\rho(g) = \langle u(g), X_g^0 \rangle.$$

In Minkowski space, $\alpha_\rho(g)$ is the *signed Lorentzian length* of a closed geodesic in $\mathbb{M}^{2,1}/\langle \rho(g) \rangle$ [20, 21].

As a function of word length in the group Γ_0 , normalized α_ρ behaves better than non-normalized $\tilde{\alpha}_\rho$. Nonetheless, the sign of $\tilde{\alpha}_\rho(g)$ is well defined and is equal to that of $\alpha_\rho(g)$. So we may extend the definition of α_ρ to parabolic g , for instance by declaring that $F_g = X_g^0$.

THEOREM 2.2. [6] *Let Γ_0 be a Fuchsian group whose corresponding hyperbolic surface Σ is homeomorphic to a three-holed sphere. Denote the generators of Γ_0 corresponding to the three components of $\partial\Sigma$ by $\partial_1, \partial_2, \partial_3$. Let ρ be an affine deformation of Γ_0 .*

If $\alpha_\rho(\partial_i)$ is positive (respectively, negative, nonnegative, nonpositive) for each i then for all $\gamma \in \Gamma_0 \setminus \{1\}$, $\alpha_\rho(\gamma)$ is positive (respectively, negative, nonnegative, nonpositive).

The proof of Theorem 2.2 relies upon showing that the affine deformation ρ of the Fuchsian group Γ_0 acts properly on $\mathbb{M}^{2,1}$. By a fundamental lemma due to Margulis [20, 21] and extended in [4], if ρ is proper, then α_ρ applied to every element has the same sign. Moreover,

- if $\alpha_\rho(\partial_1) = 0$ and $\alpha_\rho(\partial_2), \alpha_\rho(\partial_3) > 0$ then specifically $\alpha_\rho(\gamma) = 0$ only if $\gamma \in \langle \partial_1 \rangle$, and
- if $\alpha_\rho(\partial_1) = \alpha_\rho(\partial_2) = 0$ and $\alpha_\rho(\partial_3) > 0$ then specifically $\alpha_\rho(\gamma) = 0$ only if $\gamma \in \langle \delta_1, \delta_2 \rangle$.

3. Length changes in deformations

As we pointed out in the Introduction, an *affine* deformation of a holonomy representation corresponds to an *infinitesimal deformation* of the holonomy representation, or a tangent vector to the holonomy representation. In this section, we will further explore this correspondence, relating the affine Margulis invariant to the derivative of length along a path of holonomy representations. We will then prove Theorems 1.1 and 1.3 by applying Theorem 2.2, which characterizes proper deformations in terms of the Margulis invariant, to the study of length changes along a path of holonomy representations. We will close the section with some explicit computations of first order length changes.

Let $\rho_0 : \pi_1(\Sigma) \rightarrow \Gamma_0 \subset \mathbf{G}$ be a holonomy representation and let $\rho : \Gamma_0 \rightarrow \text{Aff}(\mathbb{M}^{2,1})$ be an affine deformation of ρ_0 , with corresponding cocycle $u \in Z^1(\Gamma_0, \mathfrak{g})$.

The affine deformation ρ induces a path of holonomy representations ρ_t as follows:

$$\begin{aligned}\rho_t : \pi_1(\Sigma) &\longrightarrow \mathbf{G} \\ \gamma &\longmapsto \exp(tu(g))g,\end{aligned}$$

where $g = \rho_0(\gamma)$, and u is the tangent vector to this path at $t = 0$. Conversely, for any path of representations ρ_t

$$\rho_t(\gamma) = \exp(tu(g) + O(t^2))g,$$

where $u \in Z^1(\Gamma_0, \mathfrak{g})$ and $g = \rho_0(\gamma)$.

Suppose g is hyperbolic. Then the length of the corresponding closed geodesic in Σ is

$$l(g) = 2 \cosh^{-1} \left(\frac{|\mathrm{Tr}(\tilde{g})|}{2} \right),$$

where \tilde{g} is a lift of g to $\mathrm{SL}(2, \mathbb{R})$. With ρ , ρ_t as above and $\rho_0(\gamma) = g$, set:

$$l_t(\gamma) = l(\rho_t(\gamma)).$$

Since the Margulis invariant of ρ can also be seen to be a function of its corresponding cocycle u , we may write:

$$\alpha_u(g) := \alpha_\rho(g).$$

Consequently:

$$\left. \frac{d}{dt} \right|_{t=0} l_t(\gamma) = \frac{\alpha_u(g)}{2},$$

so we may interpret α_u as the change in length of an affine deformation, up to first order [18, 15].

Although $l_t(\gamma)$ is not differentiable at 0 for parabolic g ,

$$\left. \frac{d}{dt} \right|_{t=0} \frac{\sigma(g)}{2} \mathrm{Tr}(\rho_t(\gamma)) = \tilde{\alpha}_u(g).$$

Thus Theorem 1.1 simply reinterprets Theorem 2.2.

PROOF OF THEOREM 1.3. Let ρ_t , $-\epsilon \leq t \leq \epsilon$ be a path of holonomy representations. Since we assume the boundary components to be lengthening, they must have hyperbolic holonomy on $(-\epsilon, \epsilon)$.

Suppose there exists $\gamma \in \pi_1(\Sigma)$ and $T \in (-\epsilon, \epsilon)$ such that the length of $\rho_t(\gamma)$ decreases in a neighborhood of T . Let $u = u_T \in Z^1(\Gamma_0, \mathfrak{g})$ be a cocycle tangent to the path at T ; then

$$\alpha_{u_T}(\gamma) < 0.$$

Theorem 2.2 implies that for some $i = 1, 2, 3$:

$$\alpha_{u_T}(\partial_i) < 0.$$

but then the length of the corresponding end must decrease, contradicting the hypothesis. □

3.1. Deformed hyperbolic transformations. In this and the next paragraph, we explicitly compute the trace of some deformations, to understand first order length changes.

Let $g \in \mathrm{SL}(2, \mathbb{R})$ be a hyperbolic element, thus a lift of a hyperbolic isometry of \mathbb{H}^2 . Given a tangent vector in $V \in \mathfrak{sl}(2, \mathbb{R})$, consider the following two actions on $\mathrm{SL}(2, \mathbb{R})$:

$$(3.1) \quad \pi_V : g \rightarrow \exp(V) \cdot g,$$

and

$$(3.2) \quad \pi'_V : g \rightarrow g \cdot (\exp(V))^{-1} = g \cdot \exp(-V).$$

All of our quantities are conjugation-invariant. Therefore, all of our calculations reduce to a single hyperbolic element of $\mathrm{SL}(2, \mathbb{R})$,

$$g = \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix} = \exp \left(\begin{bmatrix} s & 0 \\ 0 & -s \end{bmatrix} \right)$$

whose trace is $\mathrm{Tr}(g) = 2 \cosh(s)$. The eigenvalue frame for the action of g on $\mathfrak{sl}(2, \mathbb{R})$ is

$$X_g^0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X_g^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, X_g^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} gX_g^0g^{-1} &= X_g^0, \\ gX_g^-g^{-1} &= e^{-2s}X_g^-, \\ gX_g^+g^{-1} &= e^{2s}X_g^+. \end{aligned}$$

Write the vector $V \in \mathfrak{sl}(2, \mathbb{R})$ as

$$V = aX^0(g) + bX^-(g) + cX^+(g) = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}.$$

By direct computation, the trace of the induced deformation $\pi_V(g)$ is

$$\mathrm{Tr}(\pi_V(g)) = 2 \cosh s \cosh \sqrt{a^2 + bc} + \frac{2a \sinh s \sinh \sqrt{a^2 + bc}}{\sqrt{a^2 + bc}}.$$

Observe that when $V = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$, which is equivalent to $\alpha(\gamma) = 0$:

$$\mathrm{Tr}(\pi_V(g)) = 2 \cosh(s) \cosh(\sqrt{bc})$$

Up to first order, $\mathrm{Tr}(\pi_V(g)) = 2 \cosh(s)$.

Alternatively, when $b = c = 0$:

$$\mathrm{Tr}(\pi_V(g)) = 2 \cosh(s + a)$$

whose Taylor series about $a = 0$ does not have a linear term. We assumed that $s > 0$, defining our expanding and contracting eigenvectors. As long as $a > 0$, which corresponds to $\alpha(\gamma) > 0$, the trace of the deformed element $\pi_V(g)$ is greater than the original element g .

Now consider the deformation $\pi'_V(g) = g \cdot (\exp(V))^{-1}$. When $b = c = 0$:

$$\mathrm{Tr}(\pi'_V(g)) = 2 \cosh(s - a)$$

whose Taylor series about $a = 0$ has a nonzero linear term. As long as $a > 0$, $\text{Tr}(\pi_V(g))$ is now less than the original element g . So for this deformation, a positive Margulis invariant corresponds to a decrease in trace of the original hyperbolic element.

LEMMA 3.1. *Consider a hyperbolic $g \in \text{SL}(2, \mathbb{R})$, with corresponding closed geodesic ∂ and an affine deformation represented by $V \in \mathfrak{sl}(2, \mathbb{R})$. For the actions of V on $\text{SL}(2, \mathbb{R})$ by*

- $\pi_V(g) = \exp(V) \cdot g$ then a positive value for the Margulis invariant corresponds to first order lengthening of ∂ .
- $\pi'_V(g) = g \cdot \exp(V)$ then a positive value for the Margulis invariant corresponds to first order shortening of ∂ .

3.2. Deformed parabolic transformations. As before, we are interested in quantities invariant under conjugation. Because of this, all of our calculations can be done with a very special parabolic transformation in $\text{SL}(2, \mathbb{R})$,

$$p = \begin{bmatrix} 1 & r \\ 0 & 1 \end{bmatrix} = \exp \left(\begin{bmatrix} 0 & r \\ 0 & 0 \end{bmatrix} \right)$$

where $r > 0$ and whose trace is $\text{Tr}(p) = 2$. We choose a convenient frame for the action of p on $\mathfrak{sl}(2, \mathbb{R})$:

$$X^u(g) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, X^0(g) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, X^c(g) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

The trace of the deformation of the element p by the tangent vector V described above is

$$\text{Tr}(\pi_V(p)) = 2 \cosh(\sqrt{a^2 + bc}) + \frac{cr}{\sqrt{a^2 + bc}} \sinh(\sqrt{a^2 + bc}).$$

When $\alpha(\gamma) = 0$, or equivalently when $V = \begin{bmatrix} a & b \\ 0 & -a \end{bmatrix}$, is

$$\text{Tr}(\pi_V(p)) = 2 \cosh(a).$$

Thus the trace equals 2, in terms of a , to first order.

Alternatively, when $a = b = 0$ in the expression for V ,

$$\text{Tr}(\pi_V(p)) = 2 + cr$$

which is linear and increasing in c . As long as $c > 0$, which corresponds to $\alpha(\gamma) > 0$, the trace of the deformed element $\pi_V(p)$ majorizes the trace of the original element p .

LEMMA 3.2. *Consider a parabolic $g \in \text{SL}(2, \mathbb{R})$, and an affine deformation represented by $V \in \mathfrak{sl}(2, \mathbb{R})$. For the actions of V on $\text{SL}(2, \mathbb{R})$ by*

- $\pi_V(g) = \exp(V) \cdot g$ then a positive value for the Margulis invariant corresponds to first order increase in the trace of g ;
- $\pi'_V(g) = g \cdot \exp(V)$ then a positive value for the Margulis invariant corresponds to first order decrease in the trace of g .

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