MATH 436 HOMEWORK 1 SOLUTIONS

These solutions are meant to be a grading rubric for me. They are not necessarily the most detailed or perfectly accurate. Please let me know if you encounter any mistakes.

1. Callahan 5.5

a.) Differentiating partially, we see that $x_1 = (1, 0, a)$ and $x_2 = (0, 1, b)$. Computing the cross product, we see $x_1 \times x_2 = (-a, -b, 1)$. Computing $x_i \cdot x_j$ for $i, j = 1, 2$, we see that $G = \begin{bmatrix} 1+a^2 & ab \\ ab & 1+b^2 \end{bmatrix}$. We directly compute the determinant to get $\sqrt{g} = \sqrt{1 + a^2 + b^2}$. Since $ab = 0$ iff $a = 0$ or $b = 0$, we see coordinate lines are orthogonal iff $a = 0$ or $b = 0$.

b.) Let $\partial_i f$ denote the partial derivate of f in the variable x_i. Then differentiating partially, we see that $x_1 = (1, 0, \partial_1 f)$ and $x_2 = (0, 1, \partial_2 f)$. Computing the cross product, we see $x_1 \times x_2 = (-\partial_1 f, -\partial_2 f, 1)$. Computing $x_i \cdot x_j$ for $i, j = 1, 2$, we see that $G = \begin{bmatrix} (\partial_1 f)^2 + 1 & \partial_1 f \cdot \partial_2 f \\ \partial_1 f \cdot \partial_2 f & (\partial_2 f)^2 + 1 \end{bmatrix}$. We directly compute the determinant to get $\sqrt{g} = \sqrt{1 + (\partial_1 f)^2 + (\partial_2 f)^2}$. Since $\partial_1 f \cdot \partial_2 f = 0$ iff $\partial_i f = 0$, either $i = 1$ or 2, we see coordinate lines are orthogonal as long as one partial of f vanishes.

c.) Differentiating partially, we see that $x_1 = (-r(q^2)\sin(q^1), r(q^2)\cos(q^1), 0)$ and $x_2 = (r'(q^2)\cos(q^1), r'(q^2)\sin(q^1), z'(q^2))$. Computing the cross product, we see $x_1 \times x_2 = (r(q^2)z'(q^2)\cos(q^1), r(q^2)z'(q^2)\sin(q^1), -r'(q^2)r(q^2))$. Computing $x_i \cdot x_j$ for $i, j = 1, 2$, we see that $G = \begin{bmatrix} (r(q^2))^2 & 0 \\ 0 & (r'(q^2))^2 + (z'(q^2))^2 \end{bmatrix}$. We directly compute the determinant to get $\sqrt{g} = |r(q^2)|\sqrt{(r'(q^2))^2 + (z'(q^2))^2}$. From the non-diagonal entires of G, we conclude all coordinate lines are orthongonal.

d.) e.) These are special cases of c.), and follow appropriately.

2. Callahan 5.7

Note that in the above question part e.), $\sqrt{g} = r(a + r\cos(q^2))$. Compute the area with the usual formula:

$$\begin{aligned} A &= \int_0^{2\pi} \int_{-\pi}^{\pi} \sqrt{g}\ dq^2 dq^2 \\ &= r\int_0^{2\pi} \int_{-\pi}^{\pi} (a + r\cos(q^2))dq^1 dq^2 \\ &= 4\pi^2 ar \end{aligned}$$

3. Callahan 5.8

a.) Recall that the formula for the length of a curve $\gamma : [a,b] \to M$ is given by the formula:

$$L(\gamma) = \int_a^b \sqrt{g_{ij}\frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}}dt$$

where γ^i denotes the ith component of γ. If $q^1 \equiv C$ a constant, then we see $L(\gamma) = \int_{-\pi}^{\pi} \sqrt{r^2}dt = 2\pi r$. Now, if $q^2 \equiv C$ constant, then $\int_{-\pi}^{\pi} \sqrt{(a + r\cos(q^2)^2}dt = 2\pi(a + r\cos(q^2))$.

b.) c.) are clear.

4. Callahan 5.10

a.) Taking partial derivatives, we see $x_1 = z'(q^2) + q^2 z''(q^1)$, $x_2 = z'(q^1)$. Applying the formula for the cross product, we see $x_1 \times x_2 = q^2(z''(q^1) \times z'(q^1))$. Recall that $|z'| \equiv 1$. (Computing $x_i \cdot x_j$ and **simplifying** for $i,j = 1,2$, we compute $G = \begin{bmatrix} 1 + (q^2)^2\|z''(q^1)\|^2 & 1 \\ 1 & 1 \end{bmatrix}$ Computing the determinant, we see $\sqrt{g} = |q^2|\|z''(q^1)\|$.

b.) The parametrization will be singular iff $z''(q^1) \times z'(q^1) = 0$. Computing the $\frac{d}{dt}|z'|$ from the definition and setting it equal to zero ($|z'| \equiv 1$), we see the parametrization is singular iff $z''(q^1)$ is a scalar multiple of $z'(q^1)$. At a singularity, the dimension of the parametrization decreases from 2 to 1.

c.) We compute $\frac{x_1 \times x_2}{\|(x_1, x_2)\|} = sgn(q^2)\frac{z''(q^1) \times z'(q^1)}{\|z''(q^1) \times z'(q^1)\|}$, which is independent of q^2 modulo sign.

d.) To determine orthogonality or not of coordinate lines, we compute $x_1 \cdot x_2 = |z'(q^1)|^2 + q^2 z'(q^1)z''(q^2) = 1 + 0 = 1$. This computation follows from the fact that $|z'| \equiv 1$, and in particular, $z'(q^1) \cdot z''(q^2) = 0$. Hence, coordinate lines are never orthogonal.

e.) We see that $L = \int_a^b \sqrt{g_{ij}\frac{\gamma^i}{dt}\frac{\gamma^j}{dt}} = \int_a^b \sqrt{g_{22}}dt = \int_a^b dt = (b - a)$ as desired.

5. Written 1

a.) Let ψ_N and ψ_S denote the stereographic projection from the north and south poles, respectively. Then $\psi_S(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$, $\psi_S(x,y,z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$. The inverses are given by $\psi_S^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{-1+x^2+y^2}{x^2+y^2+1}\right)$ and $\psi_N^{-1}(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{1-x^2-y^2}{x^2+y^2+1}\right)$.

b.) We compute $\psi_S \circ \psi_N^{-1}(x,y) = \psi_N \circ \psi_S^{-1}(x,y) = \frac{(x,y)}{\|(x,y)\|^2}$.

c.) A direct computation yields $D(\psi_S \circ \psi_N^{-1}) = \frac{1}{\|(x,y)\|^2} \begin{bmatrix} -x^2+y^2 & -2xy \\ -2xy & x^2-y^2 \end{bmatrix}$

d.) This is a trivial exercise in integration, though tedious. The total length in 2π, regardless of coordinate patches used for computation.

6. Written 2

Let M be an n-dimensional (smooth) manifold, and let $\pi_1 : TM \to M$ denote projection. Then $\pi_1^{-1}(\{p\}) = T_pM$. So it suffices to show the tangent space at a point can be identified with $\mathbb{R}^n$.

Recall that we define the tangent space at a point to be the set of equivalence classes of short curves passing through p, where two curves are equivalent if they have the same tangent vector at p. We can put the equivalence classes in bijection with $\mathbb{R}^n$ by having an equivalence class of short curves map to the unique tangent value of the class at p. Precisely, the mapping is $[\gamma] \mapsto \gamma'(0)$, where a curve is parametrized $\gamma : [-1,1] \to \mathbb{R}^n$ Surjectivity is clear, and injectivity follows from the equivalence relation. This gives a set isomorphism between T_pM and $\mathbb{R}^n$, as desired.

This can be extended easily to a vector space isomorphism, once we have given the space of equivalence classes of short curves the correct $\mathbb{R}$ vector space structure. This is induced from addition of curves and scalar multiplication.