MATH 436 HOMEWORK 3 SOLUTIONS

These solutions are meant to be a grading rubric for me. They are not necessarily the most detailed or perfectly accurate. Please let me know if you encounter any mistakes.

1. PROBLEM 1
For this problem, recall that \(dx_j\left(\frac{\partial}{\partial x_i}\right) = \delta_{i,j} = 1\) if \(i=j\), 0 otherwise.

a.) \(dF\left(\frac{\partial}{\partial x_1}\right) = (2x_1, e^{x_1})\)

b.) \(dF\left(\frac{\partial}{\partial x_1}\right) = 0; \ dF\left(\frac{\partial}{\partial x_2}\right) = 1.\)

c.) \(dF = (3x_1^2dx_1, 3x_2^2dx_2, 3x_3^2dx_3, x_2x_3dx_1 + x_1x_3dx_2 + x_1x_2dx_3).\)
   Some students wrote this as a 4x3 matrix, which I gave credit for as well.

d.) Yes, the map \(F: \mathbb{R}^3 \rightarrow \mathbb{R}^3\) given by \(F(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_1 + x_3)\) is a diffeomorphism. Notice that there is an associated matrix

\[
A = \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1
\end{pmatrix}
\]

with the property that \(F(x_1, x_2, x_3) = A(x_1, x_2, x_3)\), where the multiplication on the right hand side is that of a matrix on a vector. By a standard result, \(F\) is a diffeomorphism if this matrix is non-singular. The determinant of this matrix is computed to be \(\det(A) = 2 \neq 0\), so we conclude \(F\) is a diffeomorphism.

2. PROBLEM 2

a.) \(dF(dx_1^2 + dx_2^2) = (4x_1^2 + e^{2x_1})dx_1^2\)

b.) \(dF(dx_2^2) = dx_2^2;\) so, the induced metric on \(\mathbb{R}^2\) only considers the first second coordinate.

c.) \(dF(dx_1^2 + dx_2^2 + dx_3^2 + dx_1^2) = 9x_1^2dx_1^2 + 9x_2^2dx_2^2 + 9x_3^2dx_3^2 + x_2^2x_3^2dx_1^2 + 2x_1x_2x_3dx_1dx_2 + 2x_1x_2x_3dx_1dx_3 + x_1^2x_3^2dx_2^2 + 2x_1^2x_2x_3dx_2dx_3 + x_1^2x_2^2dx_3^2.\)

d.) \(dF(dx_1^2 + dx_2^2 + dx_3^2) = dx_1^2 + dx_2^2 + dx_3^2.\) This linear map preserves the metric.

3. PROBLEM 3
Write \(\gamma(t) = (\gamma_1(t), \gamma_2(t)).\) We compute:

\[\gamma_1(t) = \gamma_2(t)\]
\[ L(\gamma) = \int_0^1 \sqrt{h(\dot{\gamma}(t), \dot{\gamma}(t))} dt \]
\[ = \int_0^1 \sqrt{h \left( \gamma_1(t) \frac{\partial}{\partial x_1} + \gamma_2(t) \frac{\partial}{\partial x_2}, \gamma_1(t) \frac{\partial}{\partial x_1} + \gamma_2(t) \frac{\partial}{\partial x_2} \right) } dt \]
\[ = \int_0^1 |\gamma_2(t)| dt \]
\[ = \int_0^1 \gamma_2(t) dt \]
\[ = \gamma_2(1) - \gamma_2(0). \]

4. Problem 4

a.) We may write \( dy^i = \sum_{k=1}^n \frac{\partial y^i}{\partial x^k} dx^k \), as per the classical change of variables formula. In terms of matrices, this can be written as
\[ dy^i = \sum_{k=1}^n J_{k,i} dx^k, \]
where \( J \) is the Jacobian matrix for the change of variables \( x \mapsto y \).
b.)
\[ g_{i,j}(y(q)) = g \left( \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right) \]
\[ = g \left( \sum_{k=1}^n J^{-1}_{k,i} \frac{\partial}{\partial x^k}, \sum_{\ell=1}^n J^{-1}_{k,j} \frac{\partial}{\partial x^\ell} \right) \]
\[ = \sum_{k,\ell=1}^n J^{-1}_{k,i} g_{k,\ell}(x(q)) J^{-1}_{\ell,j}. \]
c.) The result follows directly by substitution:
\[ \sqrt{\det|g_{i,j}(y(q))|} dy^1 \wedge dy^2 \wedge \cdots \wedge dy^n = \left( \sqrt{\det \left( \sum_{k,\ell=1}^k J^{-1}_{k,i} g_{k,\ell}(x(q)) J^{-1}_{\ell,j} \right)} \det(J) dx^1 \wedge \cdots \wedge dx^n \right) \]
\[ = \sqrt{\det g_{i,j}(x(q)) \det(J^{-1}) \det(J^{-1})^T} \det(J) dx^1 \wedge \cdots \wedge dx^n \]
\[ = \sqrt{\det g_{i,j}(x(q))} dx^1 \wedge \cdots \wedge dx^n \]