

FOCK-GONCHAROV COORDINATES FOR RANK TWO LIE GROUPS

CHRISTIAN K. ZICKERT

ABSTRACT. Let G be a simply connected, simple, complex Lie group of rank 2. We give explicit Fock-Goncharov coordinates for configurations of triples and quadruples of affine flags in G . We show that the action on triples by orientation preserving permutations corresponds to explicit quiver mutations, and that the same holds for the flip (changing the diagonal in a quadrilateral). This gives explicit coordinates on higher Teichmüller space, and also coordinates for boundary-unipotent representations of 3-manifold groups. As an application, we compute the (generic) boundary-unipotent representations in $\mathrm{Sp}(4, \mathbb{C})/\langle -I \rangle$ for the figure-eight knot complement.

1. INTRODUCTION

Let G be a simply connected, semisimple, complex Lie group with adjoint group G' . For a punctured surface S with negative Euler characteristic, Fock and Goncharov [7] define a pair $(\mathcal{A}_{G,S}, \mathcal{X}_{G',S})$ of moduli spaces, which can be viewed as an “algebraic-geometric avatar of Higher Teichmüller theory” [7, p. 5]. We shall here only consider the space $\mathcal{A}_{G,S}$. The space $\mathcal{A}_{G,S}$ has a birational atlas with a chart $\mathcal{A}_{G,\mathcal{T}}$ for each ideal triangulation \mathcal{T} together with an ordering of the vertices of each triangle compatible with the orientation of S . Each chart is a complex torus, and is constructed by gluing together copies of a configuration space of triples of affine flags in general position via a gluing pattern determined by the triangulation. Fock and Goncharov show that the atlas is *positive*, i.e. that the transition functions are subtraction free rational functions. This allows one to define the space of positive points of $\mathcal{A}_{G,S}$. When $G = \mathrm{SL}(2, \mathbb{C})$ this space is Penner’s decorated Teichmüller space [20], and the positive coordinates coming from an ideal triangulation are Penner’s λ -coordinates.

Our main result is that when G is simple of rank 2, the transition functions are given by explicit quiver mutations. For this it is enough to consider a *rotation* (a cyclic change of the vertex ordering of a triangle) and a *flip* (change of the diagonal in a quadrilateral). We also give explicit algorithms for the transition functions in higher rank, and we conjecture that they are always given by quiver mutations. When $G = \mathrm{SL}(n, \mathbb{C})$ explicit quiver mutations were given by Fock and Goncharov [7, Sec. 10].

Garoufalidis, Thurston and Zickert [12] (see also [2, 4]) used the work of Fock and Goncharov to define coordinates (called *Ptolemy coordinates*) for boundary-unipotent $\mathrm{SL}(n, \mathbb{C})$ -representations of 3-manifold groups. The relations between these coordinates (called *Ptolemy relations*) are exactly the mutation relations found by Fock and Goncharov. The coordinates seem to be very efficient for concrete computations (see [6, 5] for a database). Our main results provide similar coordinates for all simply connected, simple, complex Lie groups of rank 2.

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2. STATEMENT OF RESULTS

Let G be a simply connected, simple, complex Lie group of rank 2, i.e., G is either $A_2 = \mathrm{SL}(3, \mathbb{C})$, $B_2 = \mathrm{Spin}(5, \mathbb{C}) \cong \mathrm{Sp}(4, \mathbb{C}) = C_2$ or G_2 . There is a canonical central element $s_G \in G$, which is either trivial or of order 2 (see Section 4.2). It is trivial for A_2 and G_2 , and non-trivial for B_2 and C_2 .

Fix a maximal unipotent subgroup N and let $\mathcal{A} = \mathcal{A}_G = G/N$ denote the affine flag variety of N -cosets in G . The diagonal left G action on \mathcal{A}^k does not have a geometric quotient, but if we restrict to tuples that are sufficiently generic (see Definition 5.1), there is a geometric quotient $\mathrm{Conf}_k^*(\mathcal{A})$. It is a sub-variety of the algebro-geometric quotient $\mathcal{A}^k // G$.

To each of the groups A_2 , B_2 , C_2 and G_2 we associate a weighted quiver Q_G (see Definition 3.1) of weight 1, 2, 2, and 3, respectively. We think of the graphs as being immersed in the plane (in fact, in a triangle), but the immersion only serves as a visual representation, providing a convenient labeling scheme, and is not formally part of the data.

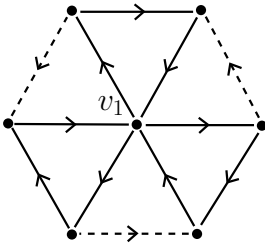


FIGURE
1. Q_{A_2} .

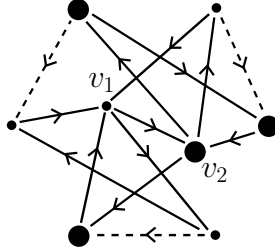


FIGURE
2. Q_{B_2} .

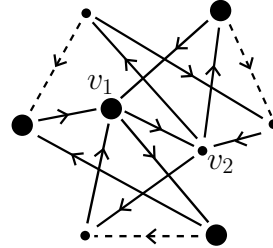


FIGURE
3. Q_{C_2} .

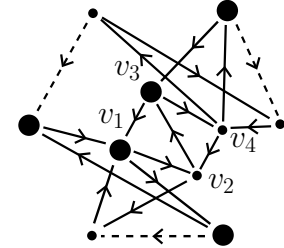


FIGURE
4. Q_{G_2} .

Every quiver Q has an associated *seed torus* T_Q (see Definition 3.7), which is a complex torus with a coordinate for each vertex. Mutation (see Definition 3.6) in a vertex v_k of Q_G gives rise to another quiver $\mu_{v_k}(Q_G)$ together with a birational map of seed tori $\mu_{v_k}: T_{Q_G} \rightarrow T_{\mu_{v_k}(Q_G)}$. For a sequence (i_1, \dots, i_k) of vertex indices define

$$(2.1) \quad \mu_{(i_1, \dots, i_k)} = \mu_{v_{i_k}} \mu_{v_{i_{k-1}}} \cdots \mu_{v_{i_1}}.$$

2.1. **Rotations.** Let

$$(2.2) \quad \mu_{A_2}^{\mathrm{rot}} = \mathrm{id}, \quad \mu_{B_2}^{\mathrm{rot}} = \mu_{C_2}^{\mathrm{rot}} = \mu_{(1,2)}, \quad \mu_{G_2}^{\mathrm{rot}} = \mu_{(1,2,3,1,4,2)}.$$

The following is a simple verification, which is illustrated in Figure 5 for $G = B_2$.

Lemma 2.1. The quiver $\mu_G^{\mathrm{rot}}(Q_G)$ is isomorphic to Q_G via an isomorphism which corresponds to a clockwise rotation by 120 degrees.

Theorem 2.2. *There is a canonical birational equivalence*

$$(2.3) \quad \mathcal{M}: \mathrm{Conf}_3^*(\mathcal{A}_G) \rightarrow T_{Q_G}$$

such that the map $(g_0N, g_1N, g_2N) \rightarrow (g_2N, g_0N, g_1N)$ corresponds to the mutation sequence μ_G under the isomorphism $T_{\mu_G^{\mathrm{rot}}(Q_G)} \cong T_{Q_G}$ induced by Lemma 2.1.

Remark 2.3. The map \mathcal{M} is the composition of a birational equivalence $\Delta: \mathrm{Conf}_3^*(\mathcal{A}_G) \rightarrow T_{Q_G}$ given by minor coordinates (see Proposition 5.9) and a monomial map $m: T_{Q_G} \rightarrow T_{Q_G}$ (see Section 8.1).

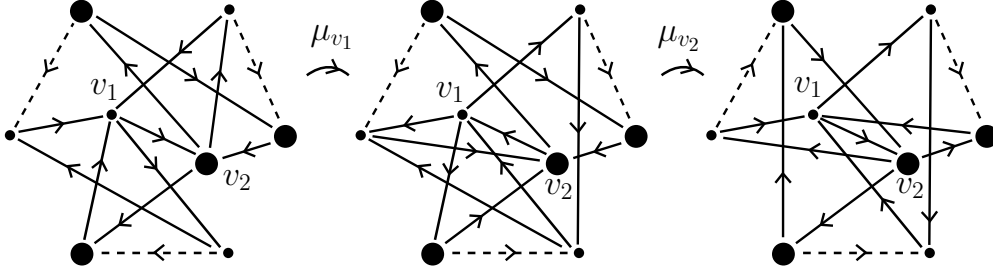


FIGURE 5. The mutation $\mu_{B_2}^{\text{rot}}$ corresponds to a rotation by 120 degrees (after rearranging the position of v_1 and v_2).

2.2. **The flip.** The generic configurations form an incomplete simplicial set with face maps

$$(2.4) \quad \varepsilon_i: \text{Conf}_k^*(\mathcal{A}) \rightarrow \text{Conf}_{k-1}^*(\mathcal{A}), \quad (g_0N, \dots, g_{k-1}N) \mapsto (g_0N, \dots, \widehat{g_iN}, \dots, g_{k-1}N).$$

For $i = 0, \dots, k$, let κ_i denote the map $\text{Conf}_k^*(\mathcal{A}) \rightarrow \text{Conf}_k^*(\mathcal{A})$ which replaces the coset g_iN by $g_i s_G N$ leaving all other cosets fixed.

2.2.1. *Gluing configurations.* We now consider configuration spaces $\text{Conf}_3^*(\mathcal{A}) \times_{02}^{s_G} \text{Conf}_3^*(\mathcal{A})$ and $\text{Conf}_3^*(\mathcal{A}) \times_{13}^{s_G} \text{Conf}_3^*(\mathcal{A})$ obtained by gluing together copies of $\text{Conf}_3^*(\mathcal{A})$ together along $\text{Conf}_2^*(\mathcal{A})$. Each is birationally equivalent to $\text{Conf}_4^*(\mathcal{A})$ and is defined by the pushout diagram

$$(2.5) \quad \begin{array}{ccccc} & & \text{Conf}_3^*(\mathcal{A}) & & \\ & \nearrow \varepsilon_i & \uparrow & \searrow \varepsilon_{j-1 \circ \kappa_{l-1}} & \\ \text{Conf}_4^*(\mathcal{A}) & \xrightarrow{\Psi_{kl}} & \text{Conf}_3^*(\mathcal{A}) \times_{kl}^{s_G} \text{Conf}_3^*(\mathcal{A}) & \dashrightarrow & \text{Conf}_2^*(\mathcal{A}) \\ & \searrow \varepsilon_j \circ \kappa_k & \downarrow & \nearrow \varepsilon_i & \\ & & \text{Conf}_3^*(\mathcal{A}) & & \end{array}$$

where (i, j, k, l) is either $(0, 2, 1, 3)$ or $(1, 3, 0, 2)$. The map Ψ_{kl} is illustrated in Figure 6.

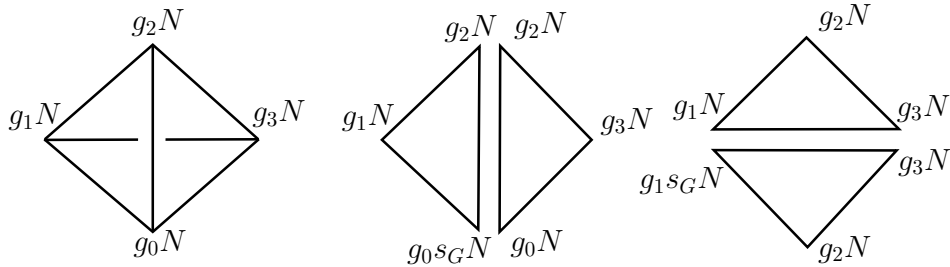


FIGURE 6. Element in $\text{Conf}_4^*(\mathcal{A})$ and its image in $\text{Conf}_3^*(\mathcal{A}) \times_{02}^{s_G} \text{Conf}_3^*(\mathcal{A})$ and $\text{Conf}_3^*(\mathcal{A}) \times_{13}^{s_G} \text{Conf}_3^*(\mathcal{A})$.

2.2.2. *Gluing quivers.* Similar to the gluing of configurations, we can glue together copies of Q_G to form quivers $Q_G \cup_{02} Q_G$ and $Q_G \cup_{13} Q_G$. The formal construction is described in Section 3.2. We denote the (non-frozen) vertices of the “right copy” of Q_G in $Q_G \cup_{02} Q_G$ by \bar{v}_i , and those of the “left copy” by v_i . Similarly, we use v_i for the “top copy” of Q_G in $Q_G \cup_{13} Q_G$ and \bar{v}_i for

the bottom copy. The two vertices on the common edge are indexed by 0 and ∞ (see Figures 7 and 8). Let

$$(2.6) \quad \begin{aligned} \mu_{A_2}^{\text{flip}} &= \mu_{(0,\infty,1,\bar{1})}, & \mu_{B_2}^{\text{flip}} &= \mu_{C_2}^{\text{flip}} = \mu_{(0,\infty,1,2,\bar{1},\bar{2},0,1,\bar{1})} \\ \mu_{G_2}^{\text{flip}} &= \mu_{(0,\infty,3,2,1,3,4,2,\bar{3},\bar{4},\bar{2},0,3,1,4,\bar{1},\bar{3},\bar{4},0,3,\bar{1},\bar{4},\bar{3},\bar{1})}. \end{aligned}$$

The following is a simple verification, which is illustrated in Figure 9 for $G = C_2$.

Lemma 2.4. There is a canonical isomorphism between $\mu_G^{\text{flip}}(Q_G \cup_{02} Q_G)$ and $Q_G \cup_{13} Q_G$.

We may thus identify the seed tori $T_{\mu_G^{\text{flip}}(Q_G \cup_{02} Q_G)}$ and $T_{Q_G \cup_{13} Q_G}$.

Remark 2.5. For verification of Lemmas 2.4 and 2.1, the java applet [14] by Mark Keller is very useful.

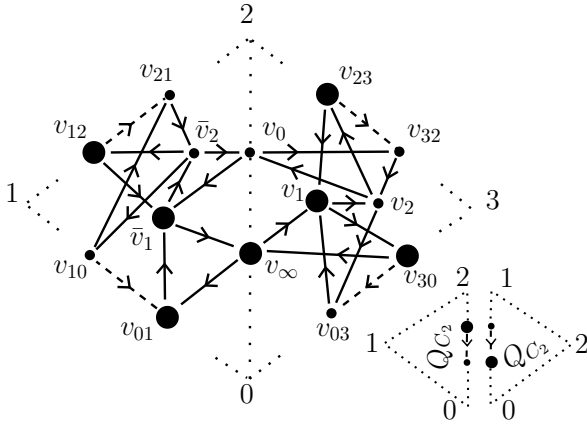


FIGURE 7. The quiver $Q_{C_2} \cup_{02} Q_{C_2}$.

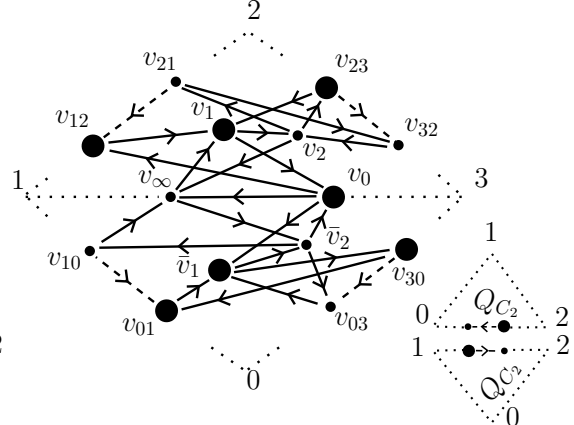


FIGURE 8. The quiver $Q_{C_2} \cup_{13} Q_{C_2}$.

Theorem 2.6. We have a commutative diagram

$$(2.7) \quad \begin{array}{ccc} \text{Conf}_3^*(\mathcal{A}) \times_{02}^{s_G} \text{Conf}_3^*(\mathcal{A}) & \xrightarrow{\cong} & T_{Q_G \cup_{02} Q_G} \\ \Psi_{13} \Psi_{02}^{-1} \downarrow & & \downarrow \mu_G^{\text{flip}} \\ \text{Conf}_3^*(\mathcal{A}) \times_{13}^{s_G} \text{Conf}_3^*(\mathcal{A}) & \xrightarrow{\cong} & T_{Q_G \cup_{13} Q_G}, \end{array}$$

where the vertical maps are induced by the map \mathcal{M} in Theorem 2.2.

Conjecture 2.7. For any semisimple, simply connected, complex Lie group G , there exists a quiver Q_G and quiver mutations μ_G^{rot} and μ_G^{flip} such that Theorems 2.2 and 2.6 hold. The map \mathcal{M} should be a composition of minor coordinates and a monomial map (see Remark 2.3).

Remark 2.8. The minor coordinates depend on a choice of reduced word for the longest element in the Weyl group. Hence, the quiver in Conjecture 2.7 should as well. In rank 2 there are only two reduced words, and we have selected the one starting with 1.

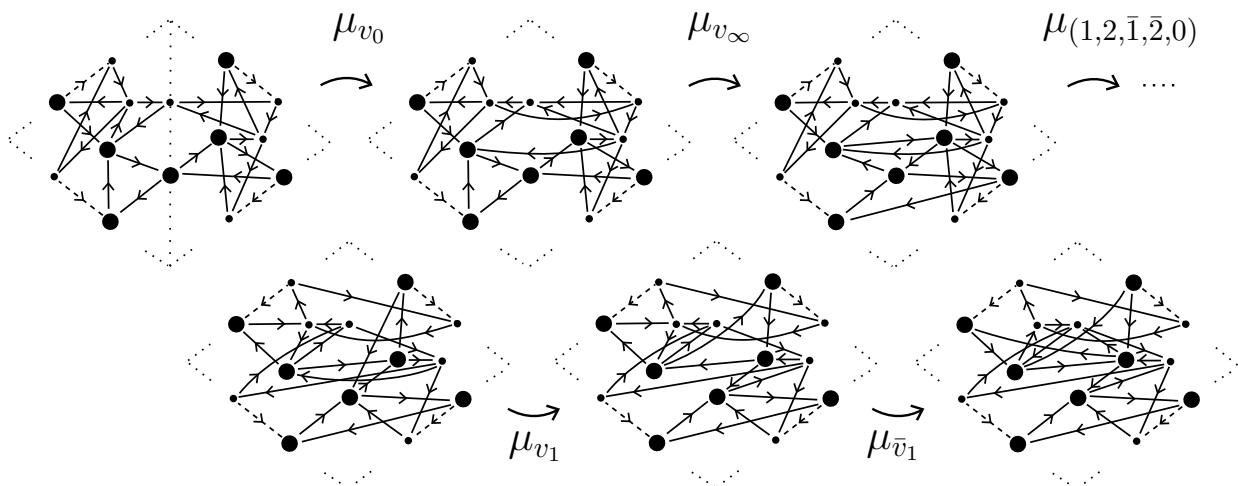


FIGURE 9. First and last two mutations in $\mu_{C_2}^{\text{flip}}(Q_{C_2})$. The final configuration corresponds to $Q_G \cup_{13} Q_G$ after rearranging the vertices inside the dotted square.

2.3. An atlas on $\mathcal{A}_{G,S}$. Let S be an oriented punctured surface with negative Euler characteristic. The universal cover of S is an open oriented disk D , and the lift of the punctures define a countable, cyclically ordered $\pi_1(S)$ -set $F_\infty(S)$ of points on $\partial\bar{D}$, the *Farey set* [7, Sec. 1.3]. Let $\bar{F}_\infty(S)$ be the double cover of $F_\infty(S)$ induced by the double cover of $\partial\bar{D}$, and let σ denote the non-trivial automorphism. The fundamental group of the punctured tangent bundle is a \mathbb{Z} -extension of $\pi_1(S)$, and the quotient by $2\mathbb{Z}$ is a central $\mathbb{Z}/2\mathbb{Z}$ -extension $\bar{\pi}_1(S)$ (see [7, Sec. 2.4]). Let $\bar{\sigma}$ denote the generator.

The space $\mathcal{A}_{G,S}$ is the moduli space of decorated twisted G -local systems on S ([7, Def 2.4]). When s_G is trivial we may regard it as the quotient stack of pairs (ρ, D) by the G -action $g(\rho, D) = (g\rho g^{-1}, gD)$, where $\rho: \pi_1(S) \rightarrow G$ is boundary-unipotent (loops encircling punctures map to conjugates of N), and $D: F_\infty(S) \rightarrow \mathcal{A}$ is a ρ -equivariant map. When s_G is non-trivial, it is the quotient stack of pairs $(\bar{\rho}, \bar{D})$, where $\bar{\rho}: \bar{\pi}_1(S) \rightarrow G$ is a boundary-unipotent representation taking $\bar{\sigma}$ to s_G , and $\bar{D}: \bar{F}_\infty(S) \rightarrow \mathcal{A}$ is $\bar{\rho}$ -equivariant (see [7, Sec. 8.6]).

Given a topological ideal triangulation \mathcal{T} of S we get an atlas on $\mathcal{A}_{G,S}$ as in [7, Sec. 8]. The process is illustrated in Figures 10 and 11 in the case when S is a twice punctured torus and $G = B_2$. Pick a fundamental polyhedron P for \mathcal{T} in D . The triangulation of S induces a triangulation of P . Pick an ordering O of the vertices of P agreeing with the cyclic ordering on $F_\infty(S)$. This associates a copy of the quiver Q_G to each triangle, and by gluing these together, we obtain a quiver whose seed torus embeds in $\mathcal{A}_{G,S}$. Note that if two edges in P are identified, the corresponding coordinates are identified as well. By [7, Thm. 8.2] this provides a positive atlas with a chart for each triple (\mathcal{T}, P, O) . Our main results give explicit formulas for how the coordinates change when changing the triple. The pair (ρ, D) , or $(\bar{\rho}, \bar{D})$, corresponding to a collection of coordinates can be explicitly computed using a *natural cocycle* (see Section 6).

2.4. 3-manifold groups and Ptolemy varieties. Let M be a compact 3-manifold with a topological ideal triangulation \mathcal{T} . A representation $\pi_1(M) \rightarrow G$ is *boundary-unipotent* if peripheral subgroups map to conjugates of N , and a *decoration* of a boundary-unipotent representation is a ρ -equivariant assignment of an N -coset to each ideal point in the universal cover of M . In Section 9.2 we define a variety $P_G(\mathcal{T})$ by gluing together configurations spaces $\text{Conf}_4^*(\mathcal{A})$ using

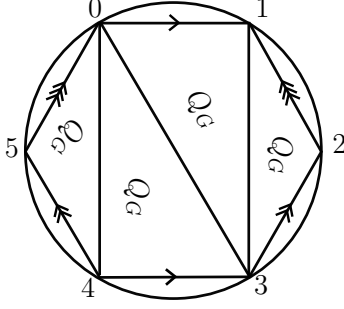


FIGURE 10. Fundamental polyhedron for the twice punctured torus S .

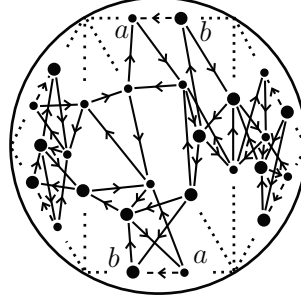


FIGURE 11. Coordinates on $A_{B_2, S}$.

a gluing pattern determined by the triangulation. Most of the results in [12] on Ptolemy varieties for $\mathrm{SL}(n, \mathbb{C})$ have natural analogues for G . As in [12, (9.26)] there is a natural one-to-one correspondence

$$(2.8) \quad \left\{ \begin{array}{c} \text{Points in} \\ P_G(\mathcal{T}) \end{array} \right\} \xleftrightarrow{1:1} \left\{ \begin{array}{c} \text{Generically decorated, boundary-unipotent} \\ \pi_1(M) \rightarrow G \end{array} \right\} / G$$

and our main results yield an explicit formula for this map.

By a result of Kostant [17] there is a canonical homomorphism $\mathrm{SL}(2, \mathbb{C}) \rightarrow G$, which preserves unipotent elements and takes $s_{\mathrm{SL}(2, \mathbb{C})}$ to s_G . If $M = \mathbb{H}^3/\Gamma$ is a cusped hyperbolic 3-manifold, there is thus a canonical boundary-unipotent representation $\rho_G: \pi_1(M) \rightarrow G/\langle s_G \rangle$. As explained e.g. in [21, 12], ρ_G need not have a boundary-unipotent lift to G , and the obstruction to the existence of such a lift is a class in $H^2(M, \partial M; \mathbb{Z}/2\mathbb{Z})$. For each $\sigma \in H^2(M, \partial M; \mathbb{Z}/2\mathbb{Z})$ there is variety $P_G^\sigma(\mathcal{T})$, and the analogue of (2.8) is (c.f. [12, (9.31)])

$$(2.9) \quad \left\{ \begin{array}{c} \text{Points in} \\ P_G^\sigma(\mathcal{T}) \end{array} \right\} \xrightarrow{z:1} \left\{ \begin{array}{c} \text{Generically decorated, boundary-unipotent} \\ \pi_1(M) \rightarrow G/\langle s_G \rangle \text{ with obstruction class } \sigma \end{array} \right\} / G,$$

where z is the order of the group $Z^1(M, \partial M; \mathbb{Z}/2\mathbb{Z})$ of $\mathbb{Z}/2\mathbb{Z}$ valued 1-cocycles (with cell structure induced by \mathcal{T}).

As in [12, Sec. 4.1], there is a natural action of H^c on $P_G(\mathcal{T})$ and $P_G^\sigma(\mathcal{T})$, where H is the maximal torus in G and c is the number of boundary components of M . The action is monomial and the quotients are denoted by $P_G(\mathcal{T})_{\mathrm{red}}$ and $P_G^\sigma(\mathcal{T})_{\mathrm{red}}$. The maps (2.8) and (2.9) induce maps

$$(2.10) \quad P_G(\mathcal{T}) \twoheadrightarrow \left\{ \begin{array}{c} \text{Boundary-unipotent} \\ \pi_1(M) \rightarrow G \end{array} \right\} / G, \quad P_G^\sigma(\mathcal{T}) \twoheadrightarrow \left\{ \begin{array}{c} \text{Boundary-unipotent} \\ \pi_1(M) \rightarrow G, \\ \text{obstruction class } \sigma \end{array} \right\} / G$$

which are generically $1 : 1$ and $|H^1(M, \partial M; \mathbb{Z}/2\mathbb{Z})| : 1$, respectively.

2.5. Computations for the figure-eight knot complement. Let $G = \mathrm{Sp}(4, \mathbb{C})$, and let M be the figure-eight knot complement with the standard ideal triangulation \mathcal{T} with 2 simplices. The fundamental group of M has a presentation

$$(2.11) \quad \pi_1(M) = \langle x_1, x_2 \mid x_1 w = w x_2, w = x_2 x_1^{-1} x_2^{-1} x_1 \rangle.$$

In Section 9.5 we show that $P_G(\mathcal{T})$ is empty and that $P_G^\sigma(\mathcal{T})$ consists of two zero-dimensional components of degree 2 and 6, respectively. The component of degree 2 is defined over $\mathbb{Q}(\sqrt{-3})$, and the corresponding representation in $\mathrm{Sp}(4, \mathbb{C})/\langle -I \rangle$ takes x_1 and x_2 to

$$(2.12) \quad \begin{bmatrix} 1 & -\frac{9(1+\sqrt{-3})}{8} & \frac{3(-1+\sqrt{-3})}{4} & 1 + \sqrt{-3} \\ 0 & 1 & -1 - \sqrt{-3} & -\frac{16}{9} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{9(1+\sqrt{-3})}{8} & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & \frac{3(-1+\sqrt{-3})}{4} & 0 \\ 0 & 0 & -2 + 2\sqrt{-3} & \frac{4(-1+\sqrt{-3})}{9} \\ \frac{1}{3} + \frac{\sqrt{-3}}{3} & -\frac{3(1+\sqrt{-3})}{2} & 8 & \frac{16}{3} \\ 0 & \frac{9}{16}(1 + \sqrt{-3}) & -9 & -4 \end{bmatrix},$$

respectively. The component of degree 6 is defined over $\mathbb{Q}(\omega)$, where

$$(2.13) \quad \omega^6 - \omega^5 + 3\omega^4 - 5\omega^3 + 8\omega^2 - 6\omega + 8 = 0,$$

and the corresponding representation is given by

$$(2.14) \quad x_1 \mapsto \begin{bmatrix} 1 & a_2 & b_1 & b_2 \\ 0 & 1 & b_3 & b_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & c_3 & 1 \end{bmatrix}, \quad x_2 \mapsto \begin{bmatrix} 0 & 0 & b'_1 & 0 \\ 0 & 0 & b'_3 & b'_4 \\ c'_1 & c'_2 & d'_1 & d'_2 \\ 0 & c'_4 & d'_3 & d'_4 \end{bmatrix},$$

where

$$(2.15) \quad \begin{aligned} a_2 = -c_3 &= \frac{\omega^5}{16} + \frac{7\omega^3}{16} - \frac{5\omega^2}{8} - \frac{5\omega}{8} - \frac{3}{2}, & b_1 &= -\frac{\omega^5}{8} + \frac{\omega^3}{8} + \frac{\omega^2}{4} - \frac{3\omega}{4} - 1, \\ b_2 = -b_3 &= -\frac{\omega^5}{32} + \frac{\omega^4}{16} + \frac{3\omega^3}{32} + \frac{\omega^2}{16} - \frac{\omega}{16} - \frac{3}{4}, & b_4 &= 2\frac{b_2}{a_2}, \\ c'_1 = -b_1^{-1} &= \frac{3\omega^5}{32} - \frac{3\omega^4}{16} + \frac{7\omega^3}{32} - \frac{11\omega^2}{16} + \frac{11\omega}{16} - \frac{1}{4}, \\ c'_2 = -\frac{\omega^5}{4} + \frac{\omega^4}{2} - \frac{3\omega^3}{4} + \omega^2 - \frac{5\omega}{2} + 3, & c'_4 = -b_2^{-1} &= -\frac{\omega^5}{4} + \frac{\omega^4}{2} - \frac{3\omega^3}{4} + \omega^2 - \frac{5\omega}{2} + 3, \\ b'_3 &= \frac{3\omega^5}{8} - \frac{\omega^4}{2} + \frac{3\omega^3}{8} - \frac{3\omega^2}{2} + \frac{9\omega}{4} - \frac{1}{2}, & b'_4 &= \frac{b'_3}{b'_1 c'_2}, \\ d'_1 &= \frac{\omega^5}{8} + \frac{3\omega^4}{4} - \frac{3\omega^3}{8} + \frac{3\omega^2}{4} - \frac{7\omega}{4} + 7, & d'_2 &= \frac{b'_4 d'_3 - b'_3 d'_4}{b'_1}, \\ d'_3 &= -\omega^5 + \omega^4 - \frac{3\omega^3}{2} + \frac{7\omega^2}{2} - 6\omega - 1, & d'_4 &= -\frac{\omega^5}{8} - \frac{3\omega^4}{4} + \frac{3\omega^3}{8} - \frac{3\omega^2}{4} + \frac{7\omega}{4} - 3. \end{aligned}$$

These representations all lift to representations in $\mathrm{Sp}(4, \mathbb{C})$, but no lift is boundary-unipotent.

Remark 2.9. We stress that the notion of genericity depends on the triangulation. There may be more representations than those detected by the Ptolemy variety. A triangulation independent Ptolemy variety detecting all irreducible representations is defined for $G = \mathrm{SL}(2, \mathbb{C})$ in [23].

3. QUIVERS, SEED TORI, AND MUTATIONS

The following definition of a (weighted) quiver serves our needs. The definition is a special case of the notion of a *seed* as defined by Fock and Goncharov [8]; see Remark 3.5. For closely related notions see e.g. [19, 15].

Definition 3.1. Let $m \geq 1$ be an integer. A *quiver* (of weight m) is a directed graph without 2-cycles together with a partition of the vertices and edges into two types; *fat vertices* (of weight

m) or not, and *half-edges or not*, respectively. In the case when $m = 1$ we do not distinguish between vertices. All edges joining two vertices are required to have the same type, and the multiplicity of a half-edge must be odd. An *isomorphism* of quivers is an isomorphism of graphs preserving the types of edges and vertices.

Example 3.2. The graphs in Figures 1, 2, 3, and 4 define quivers Q_{A_2} , Q_{B_2} , Q_{C_2} and Q_{G_2} , and we declare the weights to be 1, 2, 2, and 3, respectively.

For a quiver Q let V_Q denote the set of vertices. When $V_Q = \{v_i\}_{i \in I}$, we shall denote a vertex either by v_i or simply by i . For vertices i , and j , let σ_{ij} denote the number of directed edges from i to j counting a half-edge as $1/2$, and counting an edge from j to i negative. A quiver determines a pair of functions

$$(3.1) \quad \begin{aligned} d_Q: V_Q &\rightarrow \{1, m\}, & i &\mapsto d_i, & \varepsilon_Q: V_Q \times V_Q &\rightarrow \frac{1}{2}\mathbb{Z}, & (i, j) &\mapsto \varepsilon_{ij} \\ d_i &= \begin{cases} m & \text{if } i \text{ is fat} \\ 1 & \text{otherwise} \end{cases}, & \varepsilon_{ij} &= \frac{d_j}{\gcd(d_i, d_j)} \sigma_{ij}. \end{aligned}$$

The ε_{ij} are illustrated in Figure 12.

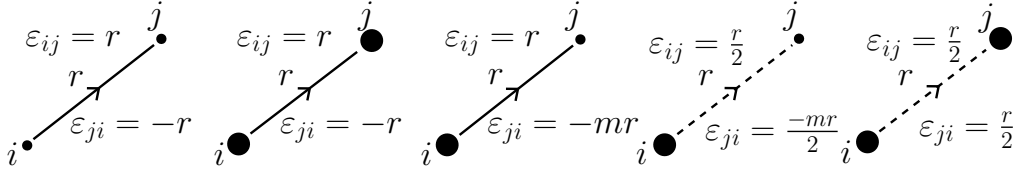


FIGURE 12. Definition of ε_{ij} when i and j are joined by an edge of multiplicity r .

Lemma 3.3. For any set V and functions $d: V \rightarrow \{1, m\}$ and $\varepsilon: V \times V \rightarrow \frac{1}{2}\mathbb{Z}$ such that $\varepsilon_{ij}/d_j = -\varepsilon_{ji}/d_i \in \mathbb{Q}$ for all $(i, j) \in V \times V$, there is a unique quiver Q with vertex set V satisfying that $d_Q = d$ and $\varepsilon_Q = \varepsilon$.

Proof. By (3.1), d determines which vertices are fat, and ε determines the multiplicity of an edge. An edge is a half-edge if and only if either ε_{ij} or ε_{ji} is a half-integer. \square

Definition 3.4. A vertex of a quiver is *frozen* if it lies on a half-edge. The set of frozen vertices is denoted by V_Q^0 .

For the quivers Q_G we index the six frozen vertices by pairs ij as shown in Figure 14.

Remark 3.5. The tuple $(V_Q, V_Q^0, \varepsilon_Q, d_Q)$ is a *seed* in the sense of [8, Def. 1.6].

3.1. Quiver mutations and seed tori. A process called *mutation* transforms one quiver to another. We follow Fock and Goncharov [8].

Definition 3.6. Let Q be a quiver and k a non-frozen vertex. Let $\mu_k(Q)$ be the unique quiver with $V_{\mu_k(Q)} = V_Q$, $d_{\mu_k(Q)} = d_Q$, and

$$(3.2) \quad \varepsilon_{\mu_k(Q)}(i, j) = \begin{cases} -\varepsilon_{ij} & \text{if } k \in \{i, j\} \\ \varepsilon_{ij} & \text{if } \varepsilon_{ik}\varepsilon_{kj} \leq 0, \quad k \notin \{i, j\} \\ \varepsilon_{ij} + |\varepsilon_{ik}|\varepsilon_{kj} & \text{if } \varepsilon_{ik}\varepsilon_{kj} > 0, \quad k \notin \{i, j\}. \end{cases}$$

We say that $\mu_k(Q)$ is obtained from Q by a *mutation* at k .

Note that mutation is an involution, i.e., $\mu_k(\mu_k(Q)) = Q$. The formula (3.2) implies that a mutation transforms the graph as shown in Figure 13. We refer to the Figures 5, 9, and 14 for examples.

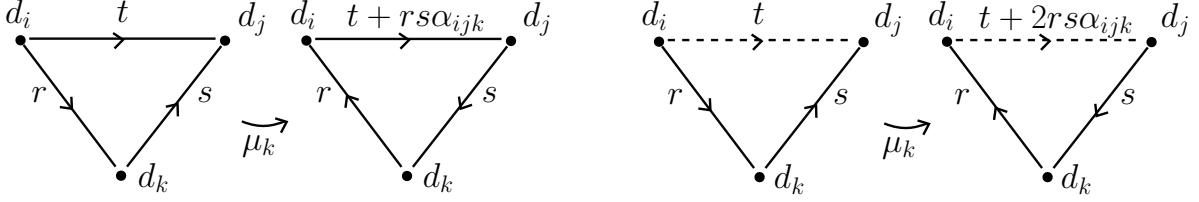


FIGURE 13. Mutation of the graph. The integer α_{ijk} is m if $d_i = d_j \neq d_k$ and 1 otherwise.

Definition 3.7. The *seed torus* associated to a quiver Q is the complex torus

$$(3.3) \quad T_Q = \text{Hom}_{\mathbb{Z}}(\Lambda_Q, \mathbb{C}^*),$$

where Λ_Q is the free abelian group generated by V_Q . The natural identification of T_Q with $(\mathbb{C}^*)^{|V_Q|}$, endows T_Q with a coordinate system $\{a_i\}_{i \in V_Q}$.

A mutation induces a birational map of seed tori

$$(3.4) \quad \mu_k: T_Q \rightarrow T_{\mu_k(Q)}$$

$$(3.5) \quad \mu_k^*(a'_k) = \frac{1}{a_k} \left(\prod_{j|\varepsilon_{kj} > 0} a_j^{\varepsilon_{kj}} + \prod_{j|\varepsilon_{kj} < 0} a_j^{-\varepsilon_{kj}} \right), \quad \mu_k^*(a'_i) = a_i, \text{ for } i \neq k$$

Since mutations are only allowed at non-frozen vertices, the coordinates of the frozen vertices always stay fixed.

Example 3.8. For the mutation shown in Figure 14 we have

$$(3.6) \quad \begin{aligned} a_{ij} &= a'_{ij} = a''_{ij}, & a'_1 &= \frac{1}{a_1} (a_{01} a_{02}^2 a_{12} + a_{20} a_2^2), & a'_2 &= a_2 \\ a''_1 &= a'_1, & a''_2 &= \frac{1}{a_2} (a'_1 a'_{10} + a'_{01} a'_{21} a'_{02}) = \frac{a_{01} a_{02}^2 a_{10} a_{12} + a_{10} a_2^2 a_{20} + a_{01} a_{02} a_1 a_{21}}{a_1 a_2}. \end{aligned}$$

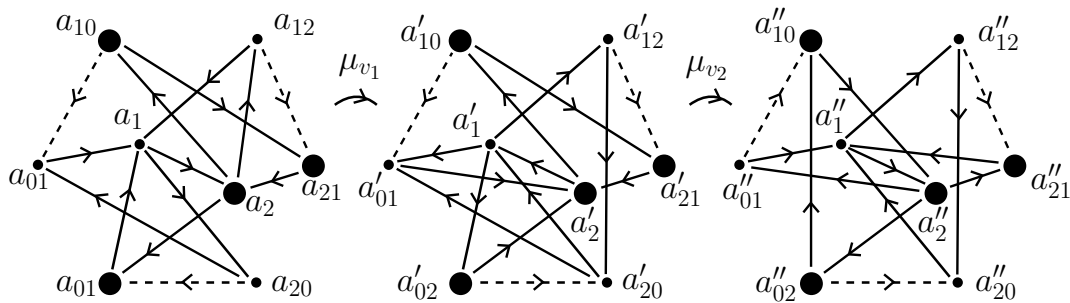


FIGURE 14. Coordinates on $T_{Q_{B_2}}$, $T_{\mu_{v_1}(Q_{B_2})}$, and $T_{\mu_{v_2}\mu_{v_1}(Q_{B_2})}$.

3.2. Gluing quivers along frozen vertices. For a subset S of V_Q , let Q_S denote the largest subgraph of Q with vertex set S . It inherits a quiver structure from Q .

Definition 3.9. Let Q and Q' be quivers and let $W \subset V_Q^0$ and $W' \subset V_{Q'}^0$ be subsets of frozen vertices, and $\phi: Q_W \rightarrow Q_{W'}$ an isomorphism. The quiver $Q \cup_\phi Q'$ is the quiver obtained by gluing together Q and Q' via ϕ , eliminating 1 and 2-cycles, and declaring that the gluing of two half-edges is a full edge (not a half-edge).

Note that the vertices in $Q \cup_\phi Q'$ corresponding to W and W' are no longer frozen, and are thus open for mutation. Also note that the seed torus for $Q \cup_\phi Q'$ is the fiber product $T_Q \times_{T_{Q_W}} T_{Q'}$.

3.2.1. The quivers $Q_G \cup_{02} Q_G$ and $Q_G \cup_{13} Q_G$. We now give a formal definition of the quivers $Q_G \cup_{02} Q_G$ and $Q_G \cup_{13} Q_G$ introduced in Section 2. Recall that the frozen vertices of Q_G are indexed by pairs ij with $i, j \in \{0, 1, 2\}$. Let $\sigma_G \in S_2$ be the trivial permutation when $G = A_2$ and the non-trivial permutation otherwise (this is the permutation of the fundamental weights given by the longest element in the Weyl group, see Section 4.1.3). Denote the non-frozen vertices of one copy of Q_G by \bar{v}_i and let

$$(3.7) \quad \phi_{02}: \{v_{01}, v_{01}\} \rightarrow \{\bar{v}_{02}, \bar{v}_{20}\}, \quad \phi_{13}: \{v_{02}, v_{20}\} \rightarrow \{\bar{v}_{12}, \bar{v}_{21}\},$$

be such that ϕ_{02} takes the pair (v_{01}, v_{01}) to $\sigma_G(\bar{v}_{02}, \bar{v}_{20})$ and ϕ_{13} takes (v_{02}, v_{20}) to $\sigma_G(\bar{v}_{12}, \bar{v}_{21})$. We can now define $Q_G \cup_{kl} Q_G$ to be $Q_G \cup_{\phi_{kl}} Q_G$. We denote the images of v_{01} and v_{10} in $Q_G \cup_{02} Q_G$ by v_∞ and v_0 , respectively. Similarly, we denote the images v_{02} and v_{20} in $Q_G \cup_{13} Q_G$ by v_∞ and v_0 . The frozen vertices of $Q_G \cup_{kl} Q_G$ are indexed according to the edges of a quadrilateral (see Figures 7 and 8).

3.3. Explicit formulas for mutations. An isomorphism of quivers induces an isomorphism of seed tori. In particular, by Lemma 2.1, we may identify $T_{\mu^{\text{rot}}(Q_G)}$ with T_{Q_G} , and the identification is such that $a'_{ij} = a_{i-1, j-1}$ (indices modulo 3).

3.3.1. Formulas for μ_G^{rot} . Following Example 3.8 the explicit formulas for the non-frozen coordinates for B_2 and C_2 are given by

$$(3.8) \quad \begin{aligned} B_2: \quad a'_1 &= \frac{1}{a_1} (a_{01} a_{02}^2 a_{12} + a_{20} a_2^2), & a'_2 &= \frac{a_{01} a_{02}^2 a_{10} a_{12} + a_{10} a_2^2 a_{20} + a_{01} a_{02} a_1 a_{21}}{a_1 a_2} \\ C_2: \quad a'_1 &= \frac{a_{01} a_{02} a_{12} + a_2 a_{20}}{a_1}, & a'_2 &= \frac{a_{10} (a_{01} a_{02} a_{12} + a_2 a_{20})^2 + a_{01}^2 a_{02} a_1^2 a_{21}}{a_1^2 a_2}. \end{aligned}$$

For G_2 the closed formula is rather lengthy so we instead introduce a “dummy variable” for each intermediate mutation.

$$(3.9) \quad \begin{aligned} G_2: \quad z_1 a_1 &= a_2 a_{20} + a_{01} a_{02} a_3, & z_2 a_2 &= a_{01}^3 a_{02}^2 a_4 + z_1^3, & a'_1 a_3 &= a_{20} a_4 + a_{12} z_1, \\ a'_2 a_4 &= a_{10} a_1^3 + a_{21} z_2, & a'_3 z_1 &= a_{01}^2 a_{02} a'_1 + z_2, & a'_4 z_2 &= a_{01}^3 a_{02} a'_2 + a_{10} a_3^3. \end{aligned}$$

3.3.2. Formulas for μ_G^{flip} . As for μ_G^{rot} we express the formulas for the non-frozen coordinates via dummy variables z_i . For A_2 the relations are *Ptolemy relations*, i.e. of the form $ef = ab + cd$.

$$(3.10) \quad \begin{aligned} A_2: \quad \bar{a}'_1 a_0 &= a_{01} a_1 + a_{03} \bar{a}_1, & a'_1 a_\infty &= a_1 a_{21} + \bar{a}_1 a_{23}, \\ a'_0 a_1 &= a_{30} a'_1 + a_{32} \bar{a}'_1, & a'_\infty \bar{a}_1 &= a_{10} a'_1 + a_{12} \bar{a}'_1. \end{aligned}$$

$$(3.11) \quad \begin{aligned} B_2 : \quad z_1 a_0 &= a_2 \bar{a}_2 + \bar{a}_1 a_{32}, & \bar{a}'_1 a_\infty &= a_{01} a_1 + \bar{a}_1 a_{30}, & z_2 a_1 &= a_{03}^2 \bar{a}_1 a_{23} + a_2^2 \bar{a}'_1, \\ \bar{a}'_2 a_2 &= a_{03} z_1 + z_2, & z_3 \bar{a}_1 &= z_1^2 + a_{12} z_2, & a'_2 \bar{a}_2 &= a_{10} a_{12} a_{32} + a_{21} z_1, \\ a'_\infty z_1 &= a'_2 \bar{a}'_2 + a_{10} z_3, & a'_0 z_2 &= a_{23} \bar{a}'_2 + \bar{a}'_1 z_3, & a'_1 z_3 &= a'_0 a_2^2 + a_{12} a_{23} a_\infty^2. \end{aligned}$$

$$(3.12) \quad \begin{aligned} C_2 : \quad z_1 a_0 &= a_2 \bar{a}_2 + \bar{a}_1^2 a_{32}, & \bar{a}'_1 a_\infty &= a_{01} a_1 + \bar{a}_1 a_{30}, & z_2 a_1 &= a_{03} \bar{a}_1 a_{23} + a_2 \bar{a}'_1, \\ \bar{a}'_2 a_2 &= a_{03} z_1 + z_2^2, & z_3 \bar{a}_1 &= z_1 + a_{12} z_2, & a'_2 \bar{a}_2 &= a_{10} a_{12}^2 a_{32} + a_{21} z_1, \\ a'_\infty z_1 &= a'_2 \bar{a}'_2 + a_{10} z_3^2, & a'_0 z_2 &= a_{23} \bar{a}'_2 + \bar{a}'_1 z_3, & a'_1 z_3 &= a'_0 a_2^2 + a_{12} a_{23} a'_\infty. \end{aligned}$$

$$(3.13) \quad \begin{aligned} G_2 : \quad z_1 a_0 &= a_{32} \bar{a}_3^3 + a_4 \bar{a}_4, & \bar{a}'_1 a_\infty &= a_{01} a_1 + a_{30} \bar{a}_3, & z_2 a_3 &= a_2 a_{23} + a_1 a_4, \\ z_3 a_2 &= a_{03} a_4^2 + z_2^3, & z_4 a_1 &= a_{03} a_{23} \bar{a}_3 + \bar{a}'_1 z_2, & z_5 z_2 &= \bar{a}_3 z_3 + a_4 z_4, \\ z_6 a_4 &= z_1 z_3 + z_5^3, & \bar{a}'_2 z_3 &= z_4^3 + a_{03} z_6, & z_7 \bar{a}_3 &= z_1 + \bar{a}_1 z_5, \\ z_8 \bar{a}_4 &= \bar{a}_1^3 a_{32} + \bar{a}_2 z_1, & a'_4 \bar{a}_2 &= a_{10} a_{12}^3 a_{32} + a_{21} z_8, & z_9 z_1 &= z_7^3 + z_6 z_8, \\ z_{10} z_5 &= z_6 + z_4 z_7, & \bar{a}'_3 z_4 &= a_{23} \bar{a}'_2 + \bar{a}'_1 z_{10}, & \bar{a}'_4 z_6 &= z_{10}^3 + \bar{a}'_2 z_9, \\ z_{11} \bar{a}_1 &= a_{12} z_7 + z_8, & z_{12} z_7 &= z_{10} z_{11} + z_9, & z_{13} z_8 &= a_{10} z_{11}^3 + a'_4 z_9, \\ a'_\infty z_9 &= a_{10} z_{12}^3 + \bar{a}'_4 z_{13}, & a'_0 z_{10} &= a_{23} \bar{a}'_4 + \bar{a}'_3 z_{12}, & z_{14} z_{11} &= a'_4 z_{12} + a_{12} z_{13}, \\ a'_2 z_{13} &= a_4^2 a'_\infty + z_{14}^3, & a'_1 z_{12} &= a_{12} a_{23} a'_\infty + a'_0 z_{14}, & a'_3 z_{14} &= a_{23} a'_2 + a'_1 a'_4. \end{aligned}$$

4. PRELIMINARIES ON LIE GROUPS

Let G be a simply connected, semisimple, complex Lie group of rank r with Lie algebra \mathfrak{g} . It is well known that G is the \mathbb{C} points of a linear algebraic group over \mathbb{Z} , and is thus an affine variety.

4.1. Basic notions. Fix a Cartan subalgebra \mathfrak{h} of \mathfrak{g} , and a set $\Pi = \{\alpha_1, \dots, \alpha_r\} \subset \mathfrak{h}^*$ of simple roots. This gives rise to a root space decomposition

$$(4.1) \quad \mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}, \quad \mathfrak{n}_- = \bigoplus_{\alpha \in \Delta_-} \mathfrak{g}_\alpha, \quad \mathfrak{n}_+ = \bigoplus_{\alpha \in \Delta_+} \mathfrak{g}_\alpha,$$

where Δ_- and Δ_+ denote the sets of negative, respectively, positive roots, and \mathfrak{g}_α denotes the root space for a root α . Let N_- , H , and N denote the Lie subgroups of G with Lie algebras \mathfrak{n}_- , \mathfrak{h} , and \mathfrak{n} , respectively. Fix Serre generators $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$, and $h_i \in \mathfrak{h}$ of \mathfrak{g} , and let

$$(4.2) \quad x_i(t) = \exp(te_i) \in N, \quad y_i(t) = \exp(tf_i) \in N_-, \quad t \in \mathbb{C}.$$

4.1.1. Fundamental weights and the Cartan matrix. Let $\langle \cdot, \cdot \rangle$ denote the symmetric bilinear form on \mathfrak{h}^* dual to the Killing form B on \mathfrak{h} . For each root α , let $H_\alpha \in \mathfrak{h}$ be the unique element satisfying that $\alpha(H) = B(H, H_\alpha)$, and let

$$(4.3) \quad \alpha^\vee = \frac{2}{\langle \alpha, \alpha \rangle} \alpha, \quad h_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} H_\alpha.$$

The element h_{α_i} is the Serre generator h_i . The set of $\gamma \in \mathfrak{h}^*$ with $\gamma(h_i) \in \mathbb{Z}$ for all i form a lattice P generated by the *fundamental weights*, which are the elements $\omega_1, \dots, \omega_r \in \mathfrak{h}^*$ satisfying that $\omega_i(h_{\alpha_j}) = \delta_{ij}$, or equivalently, that $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$. The *Cartan matrix* is the matrix A with entries $A_{ij} = \langle \alpha_i^\vee, \alpha_j \rangle$.

4.1.2. *Coordinates on H .* For every weight ω , there is a *character* $\chi_\omega: H \rightarrow \mathbb{C}^*$, and for every root α there is a *cocharacter* $\chi_\alpha^*: \mathbb{C}^* \rightarrow H$. These are defined by

$$(4.4) \quad \chi_\omega(\exp(h)) = e^{\omega(h)}, \quad \chi_\alpha^*(e^t) = \exp(h_\alpha t),$$

and satisfy

$$(4.5) \quad \chi_\omega \circ \chi_\alpha^*(t) = t^{\langle \omega, \alpha^\vee \rangle}, \quad t \in \mathbb{C}^*.$$

It follows that we have an isomorphism

$$(4.6) \quad H \cong (\mathbb{C}^*)^r, \quad h \mapsto (\chi_{\omega_1}(h), \dots, \chi_{\omega_r}(h)), \quad \chi_{\alpha_1}^*(h_1) \cdots \chi_{\alpha_r}^*(h_r) \leftarrow (h_1, \dots, h_r)$$

We may thus identify H with $(\mathbb{C}^*)^r$. We sometimes denote $\chi_{\alpha_i}^*(t)$ by h_i^t .

4.1.3. *The Weyl group and reduced words.* The *Weyl group* W is the group generated by the *simple root reflections* s_i given by

$$(4.7) \quad s_i(\gamma) = \gamma - \langle \gamma, \alpha_i^\vee \rangle \alpha_i, \quad \gamma \in \mathfrak{h}^*.$$

The Weyl group is isomorphic to $N_G(H)/H$, and there is a section (see [9, Sec. 1.4])

$$(4.8) \quad \begin{aligned} W &\rightarrow N_G(H), & w &\mapsto \bar{w} \\ \overline{s_{i_1} \cdots s_{i_k}} &\mapsto \overline{s_{i_1}} \cdots \overline{s_{i_k}}, & \overline{s_i} &= x_i(-1)y_i(1)x_i(-1). \end{aligned}$$

The Weyl group is a Coxeter group and there is a unique longest element w_0 . The Weyl group acts on \mathfrak{h}^* and permutes the simple roots and fundamental weights. It also acts on H via (4.8), i.e. $w(h) = \bar{w}h\bar{w}^{-1}$. The action by w_0 is such that $w_0(\omega_i) = -\omega_{\sigma_G(i)}$ for a permutation $\sigma_G \in S_r$. In particular, if $h = (h_1, \dots, h_r) \in H$ we have

$$(4.9) \quad w_0(h) = (h_{\sigma_G(1)}^{-1}, \dots, h_{\sigma_G(r)}^{-1}).$$

Using the explicit root data given in Section 8 one checks that $\sigma_G \in S_2$ is trivial for B_2, C_2 and G_2 and non-trivial for A_2 .

A *reduced word* for $w \in W$ is a tuple $\mathbf{i} = (i_1, \dots, i_m)$, with m minimal, such that

$$(4.10) \quad w_0 = s_{i_1} \cdots s_{i_m}.$$

In all of the following we shall fix a reduced word $\mathbf{i} = (i_0, \dots, i_m)$ for w_0 . The length m is equal to the number of positive roots.

4.2. **The element s_G .** Consider the element

$$(4.11) \quad s_G = \prod_{\alpha \in \Delta_+} \chi_\alpha^*(-1) \in H.$$

As shown in [7, Sec. 2.3], s_G is central in G and has order dividing 2, and $\overline{w_0}^{-1} = \overline{w_0}s_G$. Clearly, all coordinates of s_G are either 1 or -1 .

4.3. **Chamber weights and generalized minors.** References for this section include [1, 9, 7]. We adopt the notation of [1, 9], and warn the reader that the symbols w and ω look very similar.

Let G_0 be the Zariski open subset of elements $g \in G$ admitting a (necessarily unique) factorization

$$(4.12) \quad g = [g]_- [g]_0 [g]_+, \quad [g]_- \in N_-, \quad [g]_0 \in H, \quad [g]_+ \in N.$$

When writing $g = yhx$, we shall often implicitly assume that $y \in N_-, h \in H$, and $x \in N$. The factors y, h , and x are regular functions of $g \in G_0$.

Definition 4.1. A *chamber weight* is an element $\gamma \in \mathfrak{h}^*$ in the Weyl orbit of a fundamental weight, i.e. $\gamma = w\omega_i$ for some $i \in \{1, \dots, r\}$ and $w \in W$.

Definition 4.2. For a chamber weight $\gamma = w\omega_i$, the (*generalized*) *minor* associated to γ is the regular function $\Delta^\gamma: G \rightarrow \mathbb{C}$ whose restriction to $\overline{w}G_0$ is given by

$$(4.13) \quad \Delta^\gamma(g) = \chi_{\omega_i}([\overline{w}^{-1}g]_0) \in \mathbb{C}.$$

Remark 4.3. For $G = A_r$, $W = S_r$ and the Chamber weight for $\sigma\omega_i$ is the $i \times i$ minor with rows $\sigma(1), \dots, \sigma(i)$ and columns $1, \dots, i$.

Recall that we have fixed a reduced word $\mathbf{i} = (i_1, \dots, i_m)$ for w_0 .

Definition 4.4. An *\mathbf{i} -chamber weight* is a chamber weight of the form

$$(4.14) \quad w_k\omega_i, \quad w_k = s_{i_m}s_{i_{m-1}} \cdots s_{i_k}, \quad i \in \{1, \dots, r\}, \quad k \in \{1, \dots, m+1\}.$$

The corresponding minor is called an *\mathbf{i} -minor*.

Proposition 4.5 ([1, Prop. 2.9]). There are $m+r$ distinct \mathbf{i} -chamber weights, the fundamental weights ω_i and the weights

$$(4.15) \quad \gamma_k = w_k\omega_{i_k} = s_{i_m}s_{i_{m-1}} \cdots s_{i_k}\omega_{i_k}, \quad k \in \{1, \dots, m\}.$$

Moreover, all chamber weights $w_0\omega_i$ are \mathbf{i} -chamber weights.

Definition 4.6. We call the minors $\Delta^{w_0\omega_i}$ and Δ^{ω_i} *edge minors* and the $m-r$ remaining \mathbf{i} -minors *face minors*.

Note that the edge minors Δ^{ω_i} are the coordinates on H given in (4.6).

4.4. Some biregular isomorphisms. The *transpose map* (see e.g. [9, 1]) is the unique biregular antiautomorphism $\Psi: G \rightarrow G$ satisfying

$$(4.16) \quad \Psi(x_i(t)) = y_i(t), \quad \Psi(h) = h, \quad \Psi(y_i(t)) = x_i(t), \quad t \in \mathbb{C}, h \in H.$$

One has (see [7, p. 55])

$$(4.17) \quad \Psi(\overline{w_0}) = \overline{w_0}^{-1} = \overline{w_0}s_G.$$

The varieties $N \cap G_0\overline{w_0}$ and $N_- \cap \overline{w_0}G_0$ will be of special significance. One easily checks that Ψ restricts to a biregular isomorphism between them. Fomin and Zelevinsky [9] define a biregular isomorphism

$$(4.18) \quad \pi_-: N \cap G_0\overline{w_0} \rightarrow N_- \cap \overline{w_0}G_0, \quad x \mapsto \overline{w_0}^{-1}[x\overline{w_0}^{-1}]_+\overline{w_0}, \quad [\overline{w_0}y]_+ \leftarrow y.$$

Similarly, one has a biregular isomorphism (also considered in [18, 7])

$$(4.19) \quad \Phi: N \cap G_0\overline{w_0} \rightarrow N_- \cap \overline{w_0}G_0, \quad x \mapsto [x\overline{w_0}]_-, \quad \overline{w_0}[\overline{w_0}^{-1}y]_-\overline{w_0}^{-1} \leftarrow y.$$

Note that Φ is determined by the (equivalent) properties

$$(4.20) \quad x\overline{w_0}N = \Phi(x)[x\overline{w_0}]_0N, \quad xN_- = \Phi(x)[x\overline{w_0}]_0\overline{w_0}s_GN_-, \quad x \in N \cap G_0\overline{w_0},$$

which allow one to write a coset $x\overline{w_0}hN$ as ykN and vice versa (and similarly for N_- cosets). Each of the isomorphisms respects conjugation by elements $h \in H$, i.e., we have

$$(4.21) \quad \Psi(hxh^{-1}) = h^{-1}\Psi(x)h, \quad \Phi(hxh^{-1}) = h\Phi(x)h^{-1}, \quad \pi_-(hxh^{-1}) = h\pi_-(x)h^{-1}.$$

4.5. Factorization coordinates. Consider the map

$$(4.22) \quad x_{\mathbf{i}}: \mathbb{C}^m \rightarrow N, \quad (t_1, \dots, t_m) \mapsto x_{i_1}(t_1) \cdots x_{i_m}(t_m).$$

Theorem 4.7 below summarizes [9, Thm 2.19] and [1, Thms. 1.4, 4.3].

Theorem 4.7. *Let $x = x_{\mathbf{i}}(t_1, \dots, t_m) \in N \cap G_0 \overline{w_0}$, and let $y = \pi_-(x)$. The t_i and the \mathbf{i} -minors of y are related by the monomial expressions*

$$(4.23) \quad t_k = \frac{1}{\Delta^{w_k \omega_{i_k}}(y) \Delta^{w_{k+1} \omega_{i_k}}(y)} \prod_{j \neq i_k} \Delta^{w_k \omega_j}(y)^{-A_{j, i_k}}, \quad \Delta^{\gamma_k}(y) = \prod_{l \geq k} t_l^{\langle \gamma_k, (\alpha_l^{\mathbf{i}})^\vee \rangle},$$

where $\alpha_l^{\mathbf{i}} = w_{l+1}(\alpha_{i_l})$.

Remark 4.8. Every minor occurring is equal to either Δ^{ω_i} , or some Δ^{γ_k} . This can be seen using that $s_i \omega_j = \omega_j$ for $i \neq j$ (see e.g. [1, (2.5)]). For example, if $\mathbf{i} = (1, 2, 1, 2, 1, 2)$, then $w_3 \omega_2 = s_2 s_1 s_2 s_1 \omega_2 = s_2 s_1 s_2 \omega_2 = \gamma_4$.

Corollary 4.9. The variety $x_{\mathbf{i}}((\mathbb{C}^*)^m) \cap G_0 \overline{w_0}$ is isomorphic to the Zariski open subset of $N_- \cap \overline{w_0} G_0$ of points where the \mathbf{i} -minors are non-zero. \square

Corollary 4.10. The map $N_- \rightarrow \mathbb{C}^m$ taking y to $(\Delta^{\gamma_1}(y), \dots, \Delta^{\gamma_m}(y))$ is a birational equivalence. \square

5. CONFIGURATION SPACES OF TUPLES

Let G be as in Section 4, i.e. semisimple of rank r . Most of the results of this section can be found in Fock-Goncharov [7, Sec. 8]. Since our notation differs slightly from that of Fock and Goncharov, we give complete proofs.

Definition 5.1. A tuple $(g_0 N, \dots, g_{k-1} N) \in \mathcal{A}^k$ is *sufficiently generic* if

$$(5.1) \quad g_i^{-1} g_j \in \overline{w_0} G_0, \quad i \neq j \in \{0, \dots, k-1\},$$

a condition, which is open, and independent of the choice of coset representatives. The subvariety of \mathcal{A}^k of sufficiently generic tuples is denoted by $\mathcal{A}^{k,*}$, and the quotient of $\mathcal{A}^{k,*}$ by the diagonal left G action is denoted by $\text{Conf}_k^*(\mathcal{A})$.

It is convenient to view a tuple $(g_0 N, \dots, g_{k-1} N)$ as an ordered $(k-1)$ -simplex Δ^{k-1} together with a labeling of the i th vertex by $g_i N$.

5.1. The variety structure on $\text{Conf}_k^*(\mathcal{A})$. For $k > 2$, let \mathcal{W}_k be the Zariski open subset of $(B_- \cap \overline{w_0} G_0)^{k-2}$ consisting of points (a_2, \dots, a_{k-1}) with $a_i^{-1} a_j \in \overline{w_0} G_0$ for $i \neq j$. Let \mathcal{W}_2 be a singleton.

Proposition 5.2. For $k > 1$ we have an isomorphism of varieties

$$(5.2) \quad G \times H \times \mathcal{W}_k \rightarrow \mathcal{A}^{k,*}, \quad (g, h, a_2, \dots, a_j) \mapsto g(N, \overline{w_0} h N, a_2 N, \dots, a_{k-1} N)$$

Proof. Let $\alpha = (g_0 N, \dots, g_{k-1} N) \in \mathcal{A}^{k,*}$ with $g_i \in G$ fixed coset representatives. Since $g_i^{-1} g_j \in \overline{w_0} G_0$, we have factorizations $g_0^{-1} g_i = \overline{w_0} y_i h_i x_i$. Let

$$(5.3) \quad a_i = \overline{w_0} y_1^{-1} y_i h_i [\overline{w_0} y_1^{-1} y_i h_i]_+^{-1} \in B_- \cap \overline{G_0}, \quad i = 2, \dots, k-1.$$

The a_i and h_i are independent of the coset representatives g_i and are regular functions of α . Letting $g = g_0 \overline{w_0} y_1 \overline{w_0}^{-1}$, one has $\alpha = g(N, \overline{w_0} h_1 N, a_2 N, \dots, a_{k-1} N)$. This proves the result. \square

Corollary 5.3. The quotient $\text{Conf}_k^*(\mathcal{A}) = \mathcal{A}^{k,*}/G$ is a variety isomorphic to $H \times \mathcal{W}_k$. \square

Example 5.4. For $k = 2$ and 3 , we have

$$(5.4) \quad \begin{aligned} H &\cong \text{Conf}_2^*(\mathcal{A}), & H \times B_- \cap \overline{w_0}G_0 &\cong \text{Conf}_3^*(\mathcal{A}). \\ h &\mapsto (N, \overline{w_0}hN), & (h, a) &\mapsto (N, \overline{w_0}hN, aN). \end{aligned}$$

Note that $(g_0N, g_1N) \in \text{Conf}_2^*(\mathcal{A})$ corresponds to $[\overline{w_0}^{-1}g_0^{-1}g_1]_0 \in H$.

Definition 5.5. The representative of $\alpha \in \text{Conf}_k(\mathcal{A})$ of the form $(N, \overline{w_0}hN, a_1N, \dots, a_{k-2}N)$ is called the *canonical representative*.

5.2. Edge coordinates. We have regular maps

$$(5.5) \quad \pi_{ij}: \text{Conf}_k^*(\mathcal{A}) \rightarrow H, \quad (g_0N, \dots, g_{k-1}N) \mapsto [\overline{w_0}^{-1}g_i^{-1}g_j]_0, \quad i \neq j.$$

Note that under the isomorphism $H \cong \text{Conf}_2^*(\mathcal{A})$, π_{ij} takes $(g_0N, \dots, g_{k-1}N)$ to (g_iN, g_jN) . Since $H \cong (\mathbb{C}^*)^r$, a configuration thus gives rise to r coordinates for each edge (see Figure 15) given by the edge minors Δ^{w_i} . The following simple, but important, result illustrates the significance of the element s_G .

Lemma 5.6. Let $\alpha \in \text{Conf}_k(\mathcal{A})$. If $\pi_{ij}(\alpha) = h$ then $\pi_{ji}(\alpha) = w_0(h^{-1})s_G$.

Proof. If $(g_iN, g_jN) = (N, \overline{w_0}hN)$, then $(g_jN, g_iN) = (\overline{w_0}hN, N) = (N, \overline{w_0}kN)$, where $k \in H$ equals $[\overline{w_0}^{-1}(\overline{w_0}h)^{-1}]_0 = w_0(h^{-1})s_G$. This proves the result. \square

By (4.9) this shows that when changing the orientation of an edge, the edge coordinates are permuted and multiplied by a sign (see Figure 16).

Lemma 5.7. Let $\alpha \in \text{Conf}_3(\mathcal{A})$ and let $h_1 = \pi_{01}(\alpha)$, $h_2 = \pi_{12}(\alpha)$, and $h_3 = \pi_{20}(\alpha)$. The canonical representative of α equals

$$(5.6) \quad (N, \overline{w_0}h_1N, uw_0(h_1)h_2s_GN),$$

where u is an element in N_- satisfying that $[\overline{w_0}^{-1}u]_0 = (w_0(h_3h_1)h_2)^{-1}$.

Proof. The canonical representative has the form $(N, \overline{w_0}hN, ukN)$ for some $h, k \in H$, $u \in N_-$. By (5.5), we have $h_1 = [\overline{w_0}^{-1}\overline{w_0}h]_0 = h$ and $h_2 = [\overline{w_0}^{-1}(\overline{w_0}h)^{-1}uk]_0 = w_0(h^{-1})ks_G$, which together imply that $k = w_0(h_1)h_2s_G$, proving the first statement. For the second statement, Lemma 5.6 implies that $w_0(h_3^{-1})s_G = \pi_{02}(\alpha) = [\overline{w_0}^{-1}uw_0(h_1)h_2s_G]_0$, and it follows that $[\overline{w_0}^{-1}u]_0 = (w_0(h_3h_1)h_2)^{-1}$ as desired. \square

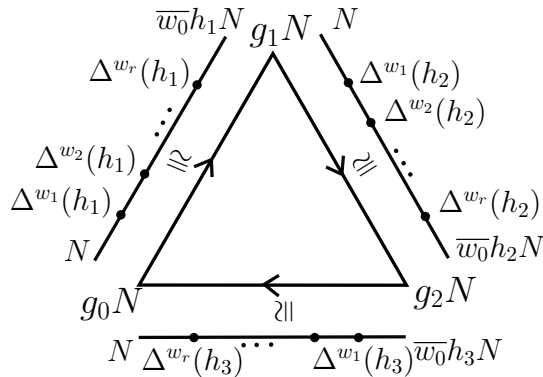


FIGURE 15. Edge coordinates.

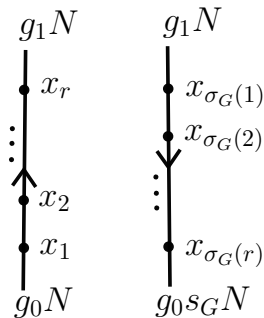


FIGURE 16. Changing the orientation of an edge.

5.3. **Face coordinates.** Consider the maps

$$(5.7) \quad H^3 \rightarrow H, \quad (h_1, h_2, h_3) \rightarrow (w_0(h_3 h_1) h_2)^{-1}, \quad N_- \cap \overline{w_0} G_0 \rightarrow H, \quad u \mapsto [\overline{w_0}^{-1} u]_0.$$

The following is a restatement of Lemma 5.7.

Lemma 5.8. We have an isomorphism of varieties

$$(5.8) \quad \text{Conf}_3(\mathcal{A}) \rightarrow H^3 \times_H N_- \cap \overline{w_0} G_0, \quad \alpha \mapsto (\pi_{01}(\alpha), \pi_{12}(\alpha), \pi_{20}(\alpha), \pi_{N_-}(\alpha)),$$

where π_{N_-} is the map $\text{Conf}_3(\mathcal{A}) \rightarrow N_- \cap \overline{w_0} G_0$ induced by the projection $B_- = N_- H \rightarrow N_-$, and \times_H denotes the fiber product with respect to the maps (5.7). \square

By Proposition 4.5 there exist $j_1 < \dots < j_{m-r} \in \{1, \dots, m\}$ such that the face minors are $\Delta^{\gamma_{j_1}}, \dots, \Delta^{\gamma_{j_{m-r}}}$. We let $\Delta^{\circ \gamma_k}$ denote the face minor $\Delta^{\gamma_{j_k}}$.

Proposition 5.9. The edge and face minors define a birational equivalence

$$(5.9) \quad \begin{aligned} \Delta: \text{Conf}_3^*(\mathcal{A}) \cong H^3 \times_H N_- \cap \overline{w_0} G_0 &\rightarrow (\mathbb{C}^*)^{3r} \times (\mathbb{C}^*)^{m-r}, \\ (h_1, h_2, h_3, u) &\mapsto (\{\Delta^{w_i}(h_1)\}_{i=1}^r, \{\Delta^{w_i}(h_2)\}_{i=1}^r, \{\Delta^{w_i}(h_3)\}_{i=1}^r, \{\Delta^{\circ \gamma_i}(u)\}_{i=1}^{m-r}). \end{aligned}$$

Proof. By definition, the edge minors $\Delta^{w_0 w_i}$ of u are the coordinates of $[\overline{w_0}^{-1} u]_0$, which by Lemma 5.7 are rational functions of h_1, h_2 and h_3 . The result now follows from Corollary 4.10. \square

By Lemma 5.6, Δ also defines a birational equivalence (also denoted by Δ)

$$(5.10) \quad \Delta: \text{Conf}_3^*(\mathcal{A}) \times_{kl}^{s_G} \text{Conf}_3^*(\mathcal{A}) \rightarrow (\mathbb{C}^*)^{5r} \times (\mathbb{C}^*)^{2(m-r)}.$$

In Section 8 we shall identify the codomains with seed tori when G is A_2, B_2, C_2 or G_2 .

5.4. **Comparison with the Ptolemy coordinates.** Using the standard root datum (as in [16]) for $G = \text{SL}(n, \mathbb{C})$, the group H is the diagonal matrices, and N is the upper triangular matrices with 1 on the diagonal. The map $\chi_{\omega_i}: H \rightarrow \mathbb{C}^*$ takes $\text{diag}(a_1, \dots, a_n)$ to $a_1 a_2 \dots a_i$, and the element $\overline{w_0}$ is the counter diagonal matrix whose $(n+1-i, i)$ entry is $(-1)^{i-1}$.

Given a triple $(g_0 N, g_2 N, g_3 N)$ of N -cosets in $\text{SL}(n, \mathbb{C})$, there is a Ptolemy coordinate c_t for each triple $t = (t_0, t_1, t_2)$ of non-negative integers summing to n defined by

$$(5.11) \quad c_t = \det(\{g_0\}_{t_0}, \{g_1\}_{t_1}, \{g_2\}_{t_2}),$$

where $\{g\}_k$ denotes the first k column vectors of a matrix g (see [12, 10]). The Ptolemy coordinate c_t of $(N, \overline{w_0} h_1 N, u w_0(h_1) h_2 s_G N)$ is up to a sign equal to the product (undefined terms are 1) of $\chi_{\omega_{t_1}}(h_1)$, $\chi_{\omega_{t_2}}(w_0(h_1) h_2)$ and the $(t_2 \times t_2)$ minor of u given by the rows $t_0 + 1, \dots, t_0 + t_1$ and the columns $1, \dots, t_1$. Using the ‘‘standard’’ word $\mathbf{i} = t_{n-1} \dots t_2 t_1$, where $t_i = s_1 s_2 \dots s_i$, it follows from Remark 4.3 that these minors are the \mathbf{i} -minors. To summarize, the Ptolemy coordinates are up to a sign and a monomial transformation equal to the minor coordinates for \mathbf{i} .

5.5. **The action on $\text{Conf}_3^*(\mathcal{A})$ by rotations.**

Proposition 5.10. The rotation map $\text{rot}: \text{Conf}_3^*(\mathcal{A}) \rightarrow \text{Conf}_3^*(\mathcal{A})$ taking $(g_0 N, g_1 N, g_2 N)$ to $(g_2 N, g_0 N, g_1 N)$ corresponds to the map

$$(5.12) \quad (h_1, h_2, h_3, u) \mapsto (h_3, h_1, h_2, h_2^{-1} w_0(h_1)^{-1} (\Phi \circ \Psi)^2(u) w_0(h_1) h_2).$$

under the isomorphism (5.8).

The proof uses the following technical lemmas.

Lemma 5.11. For any $u \in N_- \cap \overline{w_0} G_0$, we have $[\Psi(u) \overline{w_0}]_0 = [\overline{w_0}^{-1} u]_0$.

Proof. Let $\overline{w_0}^{-1}u = yhx$. We then have $[\overline{w_0}^{-1}u]_0 = h$, and $\Psi(u) = \Psi(x)h\Psi(y)\Psi(\overline{w_0})$, from which it follows that $[\Psi(u)\overline{w_0}]_0 = h = [\overline{w_0}^{-1}u]_0$. \square

Lemma 5.12. For any $u \in N_- \cap \overline{w_0}G_0$, we have $[\overline{w_0}^{-1}\Phi\Psi(u)]_0 = [\overline{w_0}^{-1}u]_0^{-1}$.

Proof. Let $\Psi(u)\overline{w_0} = \Phi\Psi(u)hx$ for $h \in H, x \in N$. Then $[\overline{w_0}^{-1}u]_0 = [\Psi(u)\overline{w_0}]_0 = h$ and

$$(5.13) \quad \overline{w_0}^{-1}\Phi\Psi(u) = (\overline{w_0}^{-1}\Psi(u)\overline{w_0})x^{-1}h^{-1} = (\overline{w_0}^{-1}\Psi(u)\overline{w_0})h^{-1}(hx^{-1}h^{-1}).$$

Hence, $[\overline{w_0}^{-1}\Phi\Psi(u)]_0 = h^{-1}$, and the result follows. \square

Lemma 5.13. For any $u \in N_- \cap \overline{w_0}G_0$, we have

$$(5.14) \quad u^{-1}N = \Psi\Phi\Psi(u)^{-1}[\overline{w_0}^{-1}u]_0^{-1}s_G\overline{w_0}N.$$

Proof. Suppose $\Psi(u)N_- = yh\overline{w_0}N_-$. We then have,

$$(5.15) \quad u^{-1}N = (\Psi(\Psi(u)N_-))^{-1} = (\Psi(yh\overline{w_0}N_-))^{-1} = \Psi(y)^{-1}h^{-1}\overline{w_0}N.$$

By (4.20), $y = \Phi\Psi(u)$ and $h = [\Psi(u)\overline{w_0}]_0s_G = [\overline{w_0}^{-1}u]_0s_G$. This proves the result. \square

Proof of Proposition 5.10. Let $\alpha = (N, \overline{w_0}h_1N, uw_0(h_1)h_2s_GN) \in \text{Conf}_3(\mathcal{A})$. One has

$$(5.16) \quad \begin{aligned} \alpha &= (N, \overline{w_0}h_1N, uw_0(h_1)h_2s_GN) \\ &= (u^{-1}N, \overline{w_0}h_1N, w_0(h_1)h_2s_GN) \\ &= (\Psi\Phi\Psi(u)^{-1}[\overline{w_0}^{-1}u]_0^{-1}s_G\overline{w_0}N, \overline{w_0}h_1N, w_0(h_1)h_2s_GN) \\ &= ([\overline{w_0}^{-1}u]_0^{-1}s_G\overline{w_0}N, \Psi\Phi\Psi(u)\overline{w_0}h_1N, w_0(h_1)h_2s_GN) \\ &= (h_2^{-1}w_0(h_1)^{-1}[\overline{w_0}^{-1}u]_0^{-1}\overline{w_0}N, h_2^{-1}w_0(h_1)^{-1}(\Phi\Psi)^2(u)[\Psi\Phi\Psi(u)\overline{w_0}]_0h_1s_GN, N) \\ &= (\overline{w_0}h_3N, h_2^{-1}w_0(h_1)^{-1}(\Phi\Psi)^2(u)w_0(h_1)h_2w_0(h_3)h_1s_GN, N). \end{aligned}$$

The third equality follows from Lemma 5.13, the fifth from (4.20), and the last from Lemma 5.7, which together with Lemmas 5.11 and 5.12 imply that

$$(5.17) \quad [\Psi\Phi\Psi(u)\overline{w_0}]_0 = [\overline{w_0}^{-1}\Phi\Psi(u)]_0 = [\overline{w_0}^{-1}u]_0^{-1} = w_0(h_3h_1)h_2.$$

This concludes the proof. \square

5.6. $\text{Conf}_4^*(\mathcal{A})$ and the flip. For $\alpha = (g_0N, g_1N, g_2N, g_3N) \in \text{Conf}_4^*(\mathcal{A})$ let

$$(5.18) \quad \begin{aligned} \alpha_{012} &= (g_0s_GN, g_1N, g_2N), & \alpha_{023} &= (g_0N, g_2N, g_3N), \\ \alpha_{123} &= (g_1N, g_2N, g_3N), & \alpha_{013} &= (g_0N, g_1s_GN, g_2N), \end{aligned}$$

so that $\Psi_{02}(\alpha) = (\alpha_{012}, \alpha_{023})$ and $\Psi_{13}(\alpha) = (\alpha_{123}, \alpha_{013})$. We wish to relate the canonical representatives of $\Psi_{02}(\alpha)$ to those of $\Psi_{13}(\alpha)$. Let $\alpha_{120} = \text{rot}^{-1}(\alpha_{012})$ and $\alpha_{130} = \text{rot}^{-1}(\alpha_{013})$. We then have

$$(5.19) \quad \Psi_{02}(\alpha) = (\text{rot}(\alpha_{120}), \alpha_{023}), \quad \Psi_{13}(\alpha) = (\alpha_{123}, \text{rot}(\alpha_{130})).$$

Hence, by Proposition 5.10 it is enough to relate the canonical representatives of α_{120} and α_{023} to those of α_{123} and α_{130} .

Each $\alpha \in \text{Conf}_4^*(\mathcal{A})$ has a unique representative of the form $(N, yk_1N, \overline{w_0}k_2N, \Phi^{-1}(v)k_3N)$. Letting $h_{ij} = \pi_{ij}(\alpha) \in H$ it follows from Lemma 5.7 that this representative is given by

$$(5.20) \quad \alpha = (N, w_0(h_{02})w_0(h_{12})^{-1}u^{-1}N, \overline{w_0}h_{02}N, \Phi^{-1}(v)\overline{w_0}h_{03}N).$$

In particular, we have

$$(5.21) \quad \begin{aligned} \alpha_{120} &= (w_0(h_{02})w_0(h_{12})^{-1}u^{-1}N, \overline{w_0}h_{02}N, s_G N) = (N, \overline{w_0}h_{12}N, uw_0(h_{12})w_0(h_{02}^{-1})s_G N) \\ \alpha_{023} &= (N, \overline{w_0}h_{02}N, \Phi^{-1}(v)\overline{w_0}h_{03}N) = (N, \overline{w_0}h_{02}N, vw_0(h_{02})h_{23}s_G N). \end{aligned}$$

Each element in G_0 also admits a factorization xyh with $x \in N_+$, $y \in N_-$ and $h \in H$. In other words, the identity induces an isomorphism

$$(5.22) \quad \iota: N_- \times H \times N \rightarrow N \times N_- \times H.$$

Proposition 5.14. Let $k = w_0(h_{12})w_0(h_{02})^{-1} \in H$, and let $c, d \in N_-$, and $l \in H$ be elements satisfying that $\iota(u, k, \Phi^{-1}(v)) = (\Phi^{-1}(c), d, l)$. Then $l = h_{31}^{-1}h_{30}$, and we have

$$(5.23) \quad \alpha_{123} = (N, \overline{w_0}h_{12}N, cw_0(h_{12})h_{23}s_G N), \quad \alpha_{130} = (N, \overline{w_0}w_0(h_{31}^{-1})N, dh_{31}^{-1}h_{30}s_G N).$$

Proof. By left multiplication by $\Phi^{-1}(c)^{-1}uk = dl\Phi^{-1}(v)^{-1}$, we have

$$(5.24) \quad \begin{aligned} \alpha &= (N, w_0(h_{02})w_0(h_{12})^{-1}u^{-1}N, \overline{w_0}h_{02}N, \Phi^{-1}(v)\overline{w_0}h_{03}N) \\ &= (dlN, N, \Phi^{-1}(c)^{-1}\overline{w_0}h_{12}N, \overline{w_0}w_0(l)h_{03}N). \end{aligned}$$

This shows that $h_{13} = w_0(l)h_{03}$, yielding the formula for l . The formulas for α_{123} and α_{130} now follow from their definition. \square

For the groups A_2 , B_2 , C_2 and G_2 , Theorem 2.6 states that after a monomial transformation, the minor coordinates of α_{012} and α_{023} are related to those of α_{123} and α_{013} by quiver mutations. The example below shows the much simpler case $A_1 = \mathrm{SL}(2, \mathbb{C})$. The case of $\mathrm{SL}(n, \mathbb{C})$ is treated in [7, Sec. 10].

Example 5.15. For $G = \mathrm{SL}(2, \mathbb{C})$, $s_G = -I$. There are no face coordinates, and the edge coordinates π_{ij} are the Ptolemy coordinates $c_{ij} = \det(g_i \begin{pmatrix} 1 \\ 0 \end{pmatrix}, g_j \begin{pmatrix} 1 \\ 0 \end{pmatrix})$. Figure 17 shows the corresponding coordinates in $\mathrm{Conf}_3^*(\mathcal{A}) \times_{kl}^{s_G} \mathrm{Conf}_3^*(\mathcal{A})$. The Ptolemy coordinates satisfy the *Ptolemy relation* $c_{03}c_{12} + c_{01}c_{23} = c_{02}c_{13}$, which is equivalent to $c_{02}(-c_{13}) = c_{23}(-c_{01}) + c_{12}(-c_{03})$, the mutation relation arising from a mutation at the middle vertex of the quiver shown on the left in Figure 17.

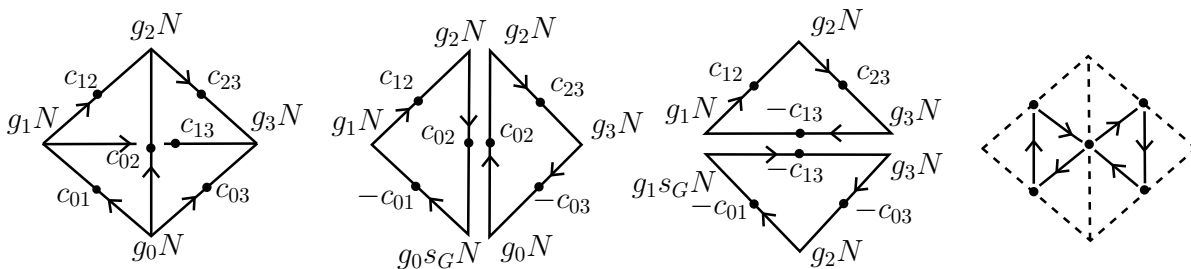


FIGURE 17. Ptolemy coordinates of a tuple and edge coordinates of its images in $\mathrm{Conf}_3^*(\mathcal{A}) \times_{02}^{s_G} \mathrm{Conf}_3^*(\mathcal{A})$ and $\mathrm{Conf}_3^*(\mathcal{A}) \times_{13}^{s_G} \mathrm{Conf}_3^*(\mathcal{A})$.

6. THE NATURAL COCYCLE

We now show that there is an explicit one-to-one correspondence between $\mathrm{Conf}_k^*(\mathcal{A})$ and certain G -valued 1-cocycles on a truncated simplex labeling long edges by elements in $\overline{w_0}H$ and short edges by elements in $N \cap G_0\overline{w_0}$. This result allows us to explicitly recover a representation from its coordinates.

For a CW complex X let $V(X)$ denote the set of vertices of X and $E(X)$ the set of oriented edges. A G -cocycle on X is a function $\tau: E(X) \rightarrow G$ such that $\tau(\varepsilon_1)\tau(\varepsilon_2)\dots\tau(\varepsilon_l) = 1$, whenever $\varepsilon_1 \cdot \varepsilon_2 \cdot \dots \cdot \varepsilon_l$ is a contractible loop. The *coboundary* of a 0-cochain $\eta: V(X) \rightarrow G$ is the G -cocycle taking an edge from vertex v to vertex w to $\eta(v)^{-1}\eta(w)$.

Let Δ^n denote a standard n -simplex, and let $\overline{\Delta}^n$ denote the corresponding truncated simplex. Let v_{ij} denote the vertex of $\overline{\Delta}^n$ near vertex i of Δ^n on the edge between i and j of Δ^n . Each edge of $\overline{\Delta}^n$ is either *long* (from v_{ij} to v_{ji}) or *short* (from v_{ij} to v_{ik}).

Definition 6.1. A G -cocycle τ on $\overline{\Delta}^n$ is a *natural cocycle* if $\tau(\varepsilon) \in N \cap G_0\overline{w_0}$ when ε is a short edge, and $\tau(\varepsilon) \in \overline{w_0}H$, when ε is a long edge.

Convention 6.2. Given a natural cocycle on $\overline{\Delta}^n$, we denote the labeling of the short edge from v_{ij} to v_{ik} by β_{jk}^i , and the labeling of the long edge from v_{ij} to v_{ji} by α_{ij} (see Figures 18 and 19).

Definition 6.3. Let $\alpha = (g_0N, \dots, g_{k-1}N) \in \text{Conf}_k^*(\mathcal{A})$. The *natural cocycle* associated to α is the coboundary of the 0-cochain η_α taking v_{ij} to g if $(g_iN, g_jN) = g(N, \overline{w_0}hN)$ with $h \in H$.

Note that the set of natural cocycles is a variety, and that the map taking a configuration to its natural cocycle is an isomorphism. We wish to give an explicit formula for the edges.

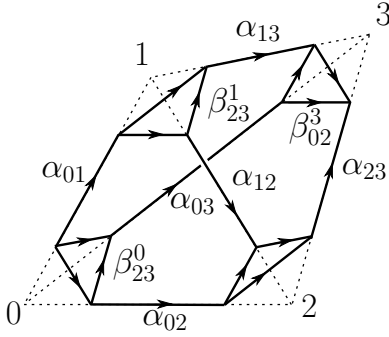


FIGURE 18. Natural cocycle on a 3-simplex.

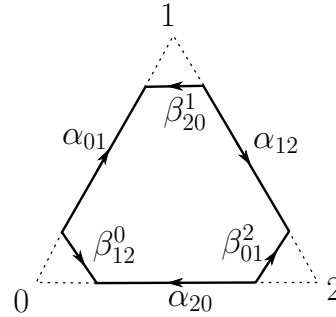


FIGURE 19. Natural cocycle on a 2-simplex.

Lemma 6.4. Let $h, k \in H$. The natural cocycle for $(N, \overline{w_0}hN, uks_GN)$ has

$$(6.1) \quad \beta_{01}^2 = k^{-1}\Psi\Phi\Psi(u)k.$$

Proof. We have $\beta_{01}^2 = \eta(v_{20})^{-1}\eta(v_{21})$, where η is the 0-cochain from Definition 6.3. Let $y = k^{-1}uk$. By Lemma 5.13, we have $y^{-1}k^{-1}s_GN = \Psi\Phi\Psi(y)^{-1}[\overline{w_0}y]_0^{-1}\overline{w_0}k^{-1}N$. Hence,

$$(6.2) \quad (uks_GN, N) = uks_G(N, y^{-1}k^{-1}s_GN) = uks_G\Psi\Phi\Psi(y)^{-1}(N, [\overline{w_0}y]_0^{-1}\overline{w_0}k^{-1}N),$$

from which it follows that $\eta(v_{20}) = uks_G\Psi\Phi\Psi(y)^{-1}$. Similarly,

$$(6.3) \quad (uks_GN, \overline{w_0}hN) = uks_G(N, \overline{w_0}hw_0(k)s_GN),$$

so $\eta(v_{21}) = uks_G$. It follows that $\beta_{01}^2 = \eta(v_{20})^{-1}\eta(v_{21}) = \Psi\Phi\Psi(y) = k^{-1}\Psi\Phi\Psi(u)k$. \square

Proposition 6.5. Let $\alpha = (h_1, h_2, h_3, u) \in H^3 \times_H N_- \cap \overline{w_0}G_0 = \text{Conf}_3^*(\mathcal{A})$, and let

$$(6.4) \quad u_0 = u, \quad u_1 = h_2^{-1}w_0(h_1)^{-1}(\Phi\Psi)^2(u_0)w_0(h_1)h_2, \quad u_2 = h_1^{-1}w_0(h_3)^{-1}(\Phi\Psi)^2(u_1)w_0(h_3)h_1.$$

The natural cocycle for α is given by

$$(6.5) \quad \begin{aligned} \beta_{01}^2 &= h_2^{-1}w_0(h_1)^{-1}\Psi\Phi\Psi(u_0)w_0(h_1)h_2, & \beta_{20}^1 &= h_1^{-1}w_0(h_3)^{-1}\Psi\Phi\Psi(u_1)w_0(h_3)h_1 \\ \beta_{12}^0 &= h_3^{-1}w_0(h_2)^{-1}\Psi\Phi\Psi(u_2)w_0(h_2)h_3, & \alpha_{01} &= \overline{w_0}h_1, \quad \alpha_{12} = \overline{w_0}h_2, \quad \alpha_{20} = \overline{w_0}h_3. \end{aligned}$$

Proof. The formula for the long edges α_{ij} is an immediate consequence of the definition, and the formula for the short edges β_{jk}^i follow from Lemma 6.4 and Proposition 5.10. \square

7. DERIVING EXPLICIT FORMULAS

We now derive formulas for the rotation (Proposition 5.10) and the flip (Proposition 5.14) in terms of the minor coordinates. Explicit computations are given for the rank two groups in Section 8. Let N_-^{\neq} denote the open subset of $N_- \cap \overline{w_0}G_0$ with non-vanishing \mathbf{i} -minors, and let

$$(7.1) \quad N^{\mathbf{i}} = x_{\mathbf{i}}((\mathbb{C}^*)^m) \cap G_0\overline{w_0}, \quad N_{\mathbf{i}}^{\bar{\mathbf{i}}} = y_{\bar{\mathbf{i}}}((\mathbb{C}^*)^m) \cap \overline{w_0}G_0,$$

where $y_{\bar{\mathbf{i}}}(s_1, \dots, s_m) = y_{i_m}(s_1)y_{i_{m-1}} \cdots y_{i_1}(s_m)$. Note that the factorization of elements in $N_{\mathbf{i}}^{\bar{\mathbf{i}}}$ is with respect to the opposite word $\bar{\mathbf{i}} = s_{i_m} \cdots s_{i_2}s_{i_1}$. The factorization coordinates on $N^{\mathbf{i}}$, $N_{\mathbf{i}}^{\bar{\mathbf{i}}}$ and the \mathbf{i} -minors on N_-^{\neq} define canonical birational equivalences of each of these spaces with $(\mathbb{C}^*)^m$.

7.1. Rotations. By Proposition 5.10 we need a formula for $(\Phi\Psi)^2$ and a formula for how the minor coordinates change under conjugation. We begin with the latter.

Lemma 7.1. For any $w \in W$, $\Delta^{w\omega_i}(k^{-1}uk) = \chi_{\omega_i}(w^{-1}(k^{-1})k)\Delta^{w\omega_i}(u)$.

Proof. For $u \in \overline{w}G_0$, one easily checks that for $[\overline{w}^{-1}k^{-1}uk]_0 = w^{-1}(k^{-1})k[\overline{w}^{-1}u]_0$. This proves the result. \square

To obtain a formula for $(\Phi\Psi)^2$ first observe that

$$(7.2) \quad (\Phi\Psi)^n = (\Psi\Phi\Psi)^{-1} \circ (\Psi\Phi)^n \circ \Psi\Phi\Psi, \quad n \in \mathbb{Z}.$$

The basic observation below allows us to apply Theorem 4.7 to explicitly compute $\Psi\Phi\Psi$.

Lemma 7.2. For any $u \in N_- \cap \overline{w_0}G_0$, we have $\pi_-(\Psi\Phi\Psi(u)) = u$.

Proof. Let $\overline{w_0}^{-1}u = yhx$. Then $x = \pi_-^{-1}(u)$, and $u = \overline{w_0}yhx$. Hence, $\Psi(u) = \Psi(x)h\Psi(y)\overline{w_0}s_G$, so $\Phi\Psi(u) = \Psi(x)$. This proves the result. \square

Corollary 7.3. The map $\Psi\Phi\Psi$ extends to a biregular isomorphism $N_-^{\neq} \rightarrow N^{\mathbf{i}}$ given explicitly by (4.23). \square .

Remark 7.4. By Proposition 6.5, this provides an explicit formula for the natural cocycle of $\alpha \in \text{Conf}_3(\mathcal{A})$ whenever the minor coordinates of α , $\text{rot}(\alpha)$ and $\text{rot}^2(\alpha)$ are non-zero. For $G = \text{SL}(n, \mathbb{C})$ and the ‘‘standard word’’ (see Section 5.4) this formula agrees with the one given in [12] via *diamond coordinates*.

7.2. The flip. For all $u \in N_- \cap \overline{w_0}G_0$, we have

$$(7.3) \quad u = \Psi(\Psi\Phi)^{-1}\Psi\Phi\Psi(u), \quad \Phi^{-1}(u) = (\Psi\Phi)^{-2}\Psi\Phi\Psi(u).$$

This motivates the definition of birational equivalences

$$(7.4) \quad \begin{aligned} \Gamma_1: N_-^{\neq} &\rightarrow N_{\mathbf{i}}^{\bar{\mathbf{i}}}, & \Gamma_2: N_-^{\neq} &\rightarrow N^{\mathbf{i}} \\ u &\mapsto \Psi(\Psi\Phi)^{-1}\Psi\Phi\Psi(u), & u &\mapsto (\Psi\Phi)^{-2}\Psi\Phi\Psi(u). \end{aligned}$$

Let $f_{(c,d,l)}$ denote the composition

$$(7.5) \quad N_-^\neq \times H \times N_-^\neq \xrightarrow{(\Gamma_1, \text{id}, \Gamma_2)} N_-^{\bar{1}} \times H \times N^{\mathbf{i}} \xrightarrow{\iota} N^{\mathbf{i}} \times N_-^{\bar{1}} \times H \xrightarrow{(\Gamma_2^{-1}, \Gamma_1^{-1}, \text{id})} N_-^\neq \times N_-^\neq \times H,$$

where ι is the map (5.22). Note that if $u, v, k, c, d,$ and l are as in Proposition 5.14, then $(c, d, l) = f_{(c,d,l)}(u, k, v)$. In particular, the flip is given explicitly in terms of ι and the maps $\Psi\Phi\Psi, \Psi\Phi$ and their inverses.

7.3. Formulas for $\Psi\Phi$ and ι . The maps $\Psi\Phi$ and ι can be computed explicitly using the following elementary properties (see e.g. [18]):

$$(7.6) \quad x_i(s)y_j(t) = y_j(t)x_i(s), \quad i \neq j$$

$$(7.7) \quad x_i(s)y_i(t) = y_i\left(\frac{t}{1+st}\right)h_i^{1+st}x_i\left(\frac{s}{1+st}\right), \quad y_i(s)x_i(t) = x_i\left(\frac{t}{1+st}\right)h_i^{\frac{1}{1+st}}y_i\left(\frac{s}{1+st}\right),$$

$$(7.8) \quad h_i^s y_j(t) = y_j(ts^{-A_{ij}})h_i^s, \quad h_i^s x_j(t) = x_j(ts^{A_{ij}})h_i^s$$

$$(7.9) \quad x_j(t)\bar{s}_j \bar{w}B = y_j(1/t)\bar{w}B, \quad w = s_{j_1} \cdots s_{j_k}, \quad s_j w \text{ reduced.}$$

Example 7.5. We compute $\Psi\Phi$ for the group A_2 using the word $\mathbf{i} = (1, 2, 1)$. The Cartan matrix is $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ and we have

$$(7.10) \quad \begin{aligned} x_1(a)x_2(b)x_1(c)\bar{s}_1\bar{s}_2\bar{s}_1B &= x_1(a)x_2(b)y_1(1/c)\bar{s}_2\bar{s}_1B \\ &= x_1(a)y_1(1/c)y_2(1/b)\bar{s}_1B \\ &= y_1\left(\frac{1}{a+c}\right)h_1^{1+a/c}x_1\left(\frac{ac}{a+c}\right)y_2(1/b)\bar{s}_1B \\ &= y_1\left(\frac{1}{a+c}\right)y_2\left(\frac{1}{b}(1+a/c)\right)y_1\left(\frac{c}{a(a+c)}\right)B \end{aligned}$$

proving that $\Psi\Phi(a, b, c) = \left(\frac{c}{a(a+c)}, \frac{a+c}{bc}, \frac{1}{a+c}\right)$.

Example 7.6. This toy example illustrates how to compute ι . Assume that $A_{12} = -1$.

$$(7.11) \quad \begin{aligned} y_2(a)y_1(b)x_1(c)x_2(d) &= y_2(a)x_1\left(\frac{c}{1+bc}\right)h_1^{\frac{1}{1+bc}}y_1\left(\frac{b}{1+bc}\right)x_2(d) \\ &= x_1\left(\frac{c}{1+bc}\right)y_2(a)x_2(d(1+bc))y_1(b(1+bc))h_1^{\frac{1}{1+bc}} \\ &= x_1\left(\frac{c}{1+bc}\right)x_2\left(\frac{d(1+bc)}{1+ad(1+bc)}\right)y_2(a(1+ad(1+bc))) \\ &\quad y_1\left(\frac{b(1+bc)}{1+ad(1+bc)}\right)h_1^{\frac{1}{1+bc}}h_2^{\frac{1}{1+ad(1+bc)}}. \end{aligned}$$

8. GROUPS OF RANK 2

We now compute the functions in Section 7 explicitly for the groups A_2, B_2, C_2 and G_2 . There are two reduced words: $(1, 2, 1)$ and $(2, 1, 2)$ for A_2 , $(1, 2, 1, 2)$ and $(2, 1, 2, 1)$ for B_2 , and $(1, 2, 1, 2, 1, 2)$ and $(2, 1, 2, 1, 2, 1)$ for G_2 . We shall always use the word starting with 1.

We use the root data from Knapp [16, Appendix C]. We identify \mathfrak{h}^* with \mathbb{R}^2 for B_2 and C_2 , and with $\{v \in \mathbb{R}^3 \mid \langle v, e_1 + e_2 + e_3 \rangle = 0\}$ for A_2 and G_2 . The e_i are the standard basis vectors,

and \langle, \rangle is the standard inner product.

$$(8.1) \quad \begin{array}{llll} A_2 : & \alpha_1 = e_1 - e_2, & \alpha_2 = e_2 - e_3, & \omega_1 = e_1, & \omega_2 = e_1 + e_2, \\ B_2 : & \alpha_1 = e_1 - e_2, & \alpha_2 = e_2, & \omega_1 = e_1, & \omega_2 = \frac{1}{2}(e_1 + e_2), \\ C_2 : & \alpha_1 = e_1 - e_2, & \alpha_2 = 2e_2, & \omega_1 = e_1, & \omega_2 = e_1 + e_2, \\ G_2 : & \alpha_1 = e_1 - e_2, & \alpha_2 = -2e_1 + e_2 + e_3, & \omega_1 = -e_2 + e_3, & \omega_2 = -e_1 - e_2 + 2e_3, \end{array}$$

Using this one easily verifies that $w_0(w_i) = -w_i$ for B_2, C_2 and G_2 , and that $w_0(w_i) = -w_{3-i}$ for A_2 , proving that σ_G is trivial for B_2, C_2 and G_2 , and non-trivial for A_2 . The Cartan matrices are $A_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $B_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$, $C_2 = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, and $G_2 = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, and one has

$$(8.2) \quad s_{A_2} = (1, 1), \quad s_{B_2} = (1, -1), \quad s_{C_2} = (-1, 1), \quad s_{G_2} = (1, 1)$$

under the identification (4.6) of H with $(\mathbb{C}^*)^2$.

8.0.1. *The map $\Psi\Phi\Psi$.* Using (4.7) and Remark 4.8 we obtain formulas for $\Psi\Phi\Psi$ and its inverse. Displayed below are $\Psi\Phi\Psi(u_1, \dots, u_m)$ and $(\Psi\Phi\Psi)^{-1}(t_1, \dots, t_m)$.

$$(8.3) \quad \begin{array}{ll} A_2 : & \left(\frac{u_2}{u_1 u_3}, \frac{u_3}{u_2}, \frac{1}{u_3} \right), & \left(\frac{1}{t_1 t_2}, \frac{1}{t_2 t_3}, \frac{1}{t_3} \right) \\ B_2 : & \left(\frac{u_2^2}{u_1 u_3}, \frac{u_3}{u_2 u_4}, \frac{u_4^2}{u_3}, \frac{1}{u_4} \right), & \left(\frac{1}{t_1 t_2^2 t_3}, \frac{1}{t_2 t_3 t_4}, \frac{1}{t_3 t_4^2}, \frac{1}{t_4} \right) \\ C_2 : & \left(\frac{u_2}{u_1 u_3}, \frac{u_3^2}{u_2 u_4}, \frac{u_4}{u_3}, \frac{1}{u_4} \right), & \left(\frac{1}{t_1 t_2 t_3}, \frac{1}{t_2 t_3^2 t_4}, \frac{1}{t_3 t_4}, \frac{1}{t_4} \right) \\ G_2 : & \left(\frac{u_2}{u_1 u_3}, \frac{u_3^2}{u_2 u_4}, \frac{u_4}{u_3 u_5}, \frac{u_5^3}{u_4 u_6}, \frac{u_6}{u_5}, \frac{1}{u_6} \right), & \left(\frac{1}{t_1 t_2 t_3^2 t_4 t_5}, \frac{1}{t_2 t_3^3 t_4^2 t_5^3 t_6}, \frac{1}{t_3 t_4 t_5^2 t_6}, \frac{1}{t_4 t_5^3 t_6^2}, \frac{1}{t_5 t_6}, \frac{1}{t_6} \right) \end{array}$$

8.0.2. *Formula for $\Psi\Phi$.* Using the algorithm in Section 7.1 we obtain (the displayed formulas are $\Psi\Phi(t_1, \dots, t_m)$ and $(\Psi\Phi)^{-1}(s_1, \dots, s_m)$)

$$(8.4) \quad \begin{array}{ll} A_2 : & \left(\frac{t_3}{t_1^2 + t_1 t_3}, \frac{t_1 + t_3}{t_2 t_3}, \frac{1}{t_1 + t_3} \right), & \left(\frac{1}{s_1 + s_3}, \frac{s_1 + s_3}{s_1 s_2}, \frac{s_1}{s_1 s_3 + s_3^2} \right) \\ B_2 : & \left(\frac{t_3 t_4^2}{t_1 \alpha_1}, \frac{\alpha_1}{t_2 t_3 t_4 \alpha_2}, \frac{\alpha_2^2}{\alpha_1}, \frac{1}{\alpha_2} \right), & \left(\frac{1}{\beta_1}, \frac{\beta_1}{\beta_2}, \frac{\beta_2^2}{s_1 s_2^2 s_3 \beta_1}, \frac{s_1 s_2}{s_4 \beta_2} \right), \\ & \alpha_1 = t_3 t_4^2 + t_1 (t_2 + t_4)^2, & \alpha_2 = t_2 + t_4, \quad \beta_1 = s_1 + s_3, \quad \beta_2 = s_1 s_2 + (s_1 + s_3) s_4 \\ C_2 : & \left(\frac{t_3 t_4}{t_1 \alpha_1}, \frac{\alpha_1^2}{t_2 t_3^2 t_4 \alpha_2}, \frac{\alpha_2}{\alpha_1}, \frac{1}{\alpha_2} \right), & \left(\frac{1}{\beta_1}, \frac{\beta_1^2}{\beta_2}, \frac{\beta_2}{s_1 s_2 s_3 \beta_1}, \frac{s_1^2 s_2}{s_4 \beta_2} \right), \\ & \alpha_1 = t_3 t_4 + t_1 (t_2 + t_4), & \alpha_2 = t_2 + t_4, \quad \beta_1 = s_1 + s_3, \quad \beta_2 = s_1^2 s_2 + (s_1 + s_3)^2 s_4, \end{array}$$

for A_2 , B_2 and C_2 , while for G_2 , we have

$$\begin{aligned}
\Psi\Phi(t) &= \left(\frac{t_3 t_4 t_5^2 t_6}{t_1 \alpha_1}, \frac{\alpha_1^3}{t_2 t_3^3 t_4^2 t_5^3 t_6 \alpha_2}, \frac{\alpha_2}{\alpha_1 \alpha_3}, \frac{\alpha_3^3}{\alpha_2 \alpha_4}, \frac{\alpha_4}{\alpha_3}, \frac{1}{\alpha_4} \right), \\
(\Psi\Phi)^{-1}(s) &= \left(\frac{1}{\beta_1}, \frac{\beta_1^3}{\beta_2}, \frac{\beta_2}{\beta_1 \beta_3}, \frac{\beta_3^3}{\beta_2 \beta_4}, \frac{\beta_4}{s_1 s_2 s_3^2 s_4 s_5 \beta_3}, \frac{s_1^3 s_2^2 s_3^3 s_4}{s_6 \beta_4} \right), \\
\alpha_1 &= t_4(t_1 t_2 t_3^2 + t_1 t_5^2 t_6 + t_3 t_5^2 t_6) + t_1 t_2 t_6 (t_3 + t_5)^2, \\
\alpha_2 &= t_4(t_2 t_3^3 t_4 + 2t_2 t_3^3 t_6 + 3t_2 t_3^2 t_5 t_6 + t_5^3 t_6^2) + t_2 t_6^2 (t_3 + t_5)^3, \\
\alpha_3 &= t_1 t_2 + t_1 t_4 + t_3 t_4 + t_1 t_6 + t_3 t_6 + t_5 t_6, \quad \alpha_4 = t_2 + t_4 + t_6, \\
\beta_1 &= s_1 + s_3 + s_5, \quad \beta_2 = s_6(s_1 + s_3 + s_5)^3 + s_4(s_1 + s_3)^3 + s_1^3 s_2, \\
\beta_3 &= s_1^2 s_2 (s_3 + s_5) + (s_1 + s_3)^2 s_3 s_4, \\
\beta_4 &= s_1^2 s_2 s_4 (s_1 s_2 s_3^3 + 3s_1 s_3 s_5^2 s_6 + 3s_3^2 s_5^2 s_6 + 2s_1 s_5^3 s_6 + 3s_3 s_5^3 s_6) + \\
&\quad s_1^3 s_2^2 s_6 (s_3 + s_5)^3 + s_4^2 s_5^3 s_6 (s_1 + s_3)^3.
\end{aligned} \tag{8.5}$$

8.0.3. *Formula for $k^{-1}uk$.* For $u \in N_-^\neq$ let $u_i = \Delta^{\gamma_i}(u)$ be the i th coordinate, and let k_1 and k_2 denote the coordinates of $k \in H$. Note that Δ^{γ_1} and Δ^{γ_2} are always edge minors, so we shall only need formulas for $(k^{-1}uk)_i$ for $2 < i \leq m$. These can be computed using Lemma 7.1 using the fact that $\bar{w}\chi_\alpha^*(t)\bar{w}^{-1} = \chi_{w(\alpha)}^*(t)$. We obtain

$$\begin{aligned}
A_2 : \quad & (k^{-1}uk)_3 = k_1^2/k_2 u_3 \\
B_2 : \quad & (k^{-1}uk)_3 = k_2^2 u_3, \quad (k^{-1}uk)_4 = k_2^2/k_1 u_4 \\
C_2 : \quad & (k^{-1}uk)_3 = k_2 u_3, \quad (k^{-1}uk)_4 = k_2^2/k_1^2 u_4 \\
G_2 : \quad & (k^{-1}uk)_3 = k_2 u_3, \quad (k^{-1}uk)_4 = k_2^3/k_1^3 u_4, \\
& (k^{-1}uk)_5 = k_2/k_1 u_5, \quad (k^{-1}uk)_6 = k_2^2/k_1^3 u_6.
\end{aligned} \tag{8.6}$$

8.1. **A monomial transformation.** Define a monomial transformation $m_G: T_{Q_G} \rightarrow T_{Q_G}$ as follows:

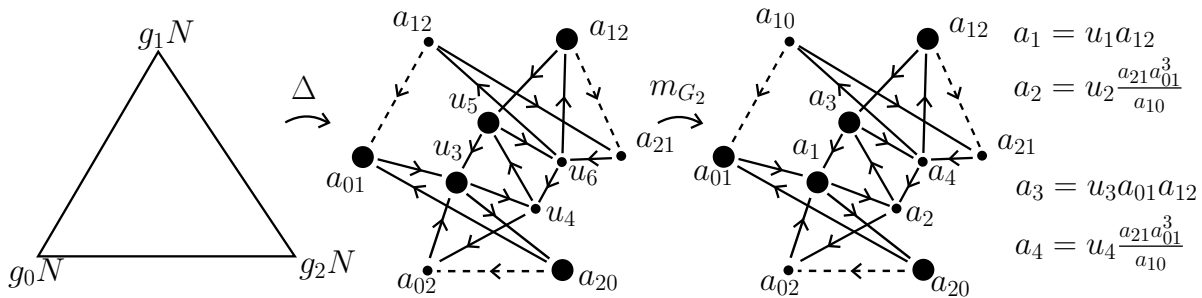
$$\begin{aligned}
m_G^*(a_{ij}) &= a_{ij}, & m_{A_2}^*(a_1) &= a_1 \frac{a_{01} a_{12}}{a_{10}} \\
(8.7) \quad m_{B_2}^*(a_1) &= a_1 a_{12}, \quad m_{B_2}^*(a_2) = a_2 \frac{a_{21}}{a_{10}}, & m_{C_2}^*(a_1) &= a_1 a_{12}, \quad m_{C_2}^*(a_2) = a_2 \frac{a_{21} a_{01}^2}{a_{10}} \\
m_{G_2}^*(a_1) &= a_1 a_{12}, \quad m_{G_2}^*(a_2) = a_2 \frac{a_{21} a_{01}^3}{a_{10}}, & m_{G_2}^*(a_3) &= a_3 a_{01} a_{12}, \quad m_{G_2}^*(a_4) = a_4 \frac{a_{21} a_{01}^3}{a_{10}}
\end{aligned}$$

We identify the codomain of the map Δ in Proposition 5.9 with the seed torus of Q_G by identifying the edge coordinates with the frozen coordinates and the face coordinates with the non-frozen coordinates. We can now define the map $\mathcal{M}: \text{Conf}_3^*(\mathcal{A}) \rightarrow T_{Q_G}$ in Theorem 2.2 to be the composition of Δ with m_G . This is illustrated in Figure 20 for $G = G_2$. Similarly, one identifies the codomain of $\Delta: \text{Conf}_3^*(\mathcal{A}) \times_{kl}^{sG} \text{Conf}_3^*(\mathcal{A}) \rightarrow (\mathbb{C}^*)^5 \times (\mathbb{C}^*)^{2(m-2)}$ with the seed torus $T_{Q_G \cup_{kl} Q_G}$ for $kl = 02$ or 13 .

8.2. **Proof of Theorem 2.2.** We wish to prove that

$$(8.8) \quad \mu_G^{\text{rot}} \mathcal{M}(h_{01}, h_{12}, h_{20}, u) = \mathcal{M}(h_{20}, h_{01}, h_{12}, (w_0(h_{01})h_{12})^{-1}(\Phi\Psi)^2(u)w_0(h_{01})h_{12})$$

for all $\alpha = (h_{01}, h_{12}, h_{20}, u) \in N_- \cap \bar{w}_0 G_0 \cong \text{Conf}_3^*(\mathcal{A})$. This is simply a matter of applying the explicit formulas above and comparing with the formula for μ_G^{rot} in Section 3.3. Clearly the

FIGURE 20. The map $\mathcal{M}: \text{Conf}_3^*(\mathcal{A}) \rightarrow T_{Q_{G_2}}$.

frozen coordinates correspond, so we only need to check the non-frozen coordinates. We do this for C_2 and leave the other groups to the reader. Let a_{ij} and a_{ji} denote the coordinates of h_{ij} . These are the frozen coordinates. By (3.8) the non-frozen coordinates of $\mu_{C_2}^{\text{rot}} \mathcal{M}(\alpha)$ are given by

$$(8.9) \quad a'_1 = \frac{a_{01}a_{02}a_{12} + a_2a_{20}}{a_1}, \quad a'_2 = \frac{a_{10}(a_{01}a_{02}a_{12} + a_2a_{20})^2 + a_{01}^2a_{02}a_1^2a_{21}}{a_1^2a_2}.$$

By Lemma 5.7 we have

$$(8.10) \quad (u_1, u_2) = [\overline{w_0}^{-1}u]_0 = (w_0(h_{20}h_{01})h_{12})^{-1} = \left(\frac{a_{20}a_{01}}{a_{12}}, \frac{a_{02}a_{10}}{a_{21}}\right)$$

and by (8.7), $a_1 = u_3a_{12}$ and $a_2 = u_4 \frac{a_{21}a_{01}^3}{a_{10}}$. Plugging these into (8.9) and using (8.10) we obtain

$$(8.11) \quad a'_1 = \frac{a_{01}a_{21}(u_2 + u_1u_4)}{a_{10}u_3}, \quad a'_2 = \frac{a_{21}(u_2^2 + u_1^2u_4^2 + u_2(u_3^2 + 2u_1u_4))}{u_3^2u_4}.$$

We now compare this to the coordinates of $\mathcal{M}(\text{rot}(\alpha))$. Using (8.3) and (8.4) we obtain

$$(8.12) \quad (\Phi\Psi)^2(u_1, u_2, u_3, u_4) = \left(u_1, u_2, \frac{u_2 + u_1u_4}{u_3}, \frac{u_2(u_2^2 + u_1^2u_4^2 + u_2(u_3^2 + 2u_1u_4))}{u_1^2u_3^2u_4}\right).$$

Hence, by (8.6) and (8.7), the non-frozen coordinates of $\mathcal{M}(\text{rot}(\alpha))$ are

$$(8.13) \quad k_2 \frac{u_2 + u_1u_4}{u_3} a_{01}, \quad \frac{k_2^2 u_2(u_2^2 + u_1^2u_4^2 + u_2(u_3^2 + 2u_1u_4))}{k_1^2 u_1^2 u_3^2 u_4} \frac{a_{10}a_{20}^2}{a_{02}},$$

where $k_1 = a_{01}^{-1}a_{12}$ and $k_2 = a_{10}^{-1}a_{21}$ are the coordinates of $k = w_0(h_{01})h_{12}$. Using (8.10) it follows that these equal a'_1 and a'_2 , respectively. This proves the result.

8.3. Proof of Theorem 2.6. Let α_{ijk} be as in Section 5.6. By Theorem 2.2 and (5.19) we must prove that

$$(8.14) \quad \mu_G^{\text{flip}}(\mu_G^{\text{rot}} \mathcal{M}(\alpha_{120}), \mathcal{M}(\alpha_{023})) = (\mathcal{M}(\alpha_{123}), \mu_G^{\text{rot}} \mathcal{M}(\alpha_{130})).$$

As in Section 5.6 we may assume that

$$(8.15) \quad \begin{aligned} \alpha_{120} &= (N, \overline{w_0}h_{12}N, uw_0(h_{12})w_0(h_{02}^{-1})s_GN), & \alpha_{023} &= (N, \overline{w_0}h_{02}N, vw_0(h_{02})h_{23}s_GN) \\ \alpha_{123} &= (N, \overline{w_0}h_{12}N, cw_0(h_{12})h_{23}s_GN), & \alpha_{130} &= (N, \overline{w_0}w_0(h_{31}^{-1})N, dh_{31}^{-1}h_{30}s_GN), \end{aligned}$$

where $(c, d, l) = f_{(c,d,l)}(u, k, v)$. As in the proof of Theorem 2.2 this is simply a matter of computing both sides of (8.14) using the explicit formulas for $\Psi\Phi$, $\Psi\Phi\Psi$ and their inverses, and the algorithm for computing ι . We give a detailed proof only for $G = C_2$. Clearly, the frozen coordinates agree, so we only need to consider the non-frozen coordinates. Let a_{ij} , a_i , \bar{a}_i , a_0 and a_∞ denote the coordinates in $T_{Q_{C_2} \cup_{02} Q_{C_2}}$ of $(\mu_{C_2}^{\text{rot}}(\mathcal{M}(\alpha_{120})), \alpha_{023})$. Note that the

coordinates of the elements h_{12} , h_{02} , and h_{23} involved in (8.15) are (a_{12}, a_{21}) , (a_∞, a_0) and (a_{23}, a_{32}) , respectively. As in Section 8.2 we have

$$(8.16) \quad \begin{aligned} \bar{a}_1 &= \frac{a_{12}a_0(u_2 + u_1u_4)}{a_{21}u_3}, & \bar{a}_2 &= a'_2 = \frac{a_0(u_2^2 + u_1^2u_4^2 + u_2(u_3^2 + 2u_1u_4))}{u_3^2u_4} \\ a_1 &= v_3a_{23}, & a_2 &= v_4 \frac{a_{32}a_\infty^2}{a_0}, \end{aligned}$$

and as in (8.10) we have

$$(8.17) \quad a_{30} = \frac{v_1a_{23}}{a_\infty}, \quad a_{03} = \frac{v_2a_{32}}{a_0}, \quad a_{01} = \frac{u_1a_\infty}{a_{12}}, \quad a_{10} = \frac{u_2a_0}{a_{21}}.$$

Using (3.12) we obtain that the face coordinates of $\mu_G^{\text{flip}}(\mu_G^{\text{rot}}\mathcal{M}(\alpha_{120}), \mathcal{M}(\alpha_{023}))$ are

$$(8.18) \quad \begin{aligned} a'_1 &= a_{23} \left(u_3 + \frac{a_{21}v_3}{a_0} + \frac{a_0a_{12}^2u_4(v_2 + v_3^2 + v_1v_4)}{a_{21}a_\infty^2v_3v_4} \right), & a'_2 &= \frac{a_{12}^2a_{32}u_4}{a_{21}} + \frac{a_{21}a_{32}a_\infty^2v_4}{a_0^2}, \\ \bar{a}'_1 &= \frac{a_0a_{12}a_{23}(u_2 + u_1u_4)v_1}{a_{21}a_\infty^2u_3} + \frac{a_{23}u_1v_3}{a_{12}}, \\ \bar{a}'_2 &= a_{32} \left(\frac{a_0a_{12}^2(u_2 + u_1u_4)^2v_2}{a_{21}^2a_\infty^2u_3^2v_4} + \frac{(u_2(u_2 + u_3^2) + 2u_1u_2u_4 + u_1^2u_4^2)v_2}{a_0u_3^2u_4} + \right. \\ &\quad \left. \frac{(a_{21}a_\infty^2u_1u_3v_3v_4 + a_0a_{12}^2(u_2 + u_1u_4)(v_2 + v_1v_4))^2}{a_0a_{12}^2a_{21}^2a_\infty^2u_3^2v_3^2v_4} \right), \end{aligned}$$

and the non-frozen edge coordinates a'_∞ and a'_0 are given by

$$(8.19) \quad \begin{aligned} a'_\infty &= a_{32} \left(\frac{v_2}{a_0^2} + \frac{a_{21}a_\infty^2(u_2 + u_3^2)v_4}{a_0^2a_{12}^2u_4} + \frac{2u_3(v_2 + v_1v_4)}{a_0v_3} + \frac{u_2}{a_{21}} + \right. \\ &\quad \left. \frac{a_{12}^2u_4(v_2(v_2 + v_3^2) + 2v_1v_2v_4 + v_1^2v_4^2)}{a_{21}a_\infty^2v_3^2v_4} \right), \\ a'_0 &= a_{23} \left(\frac{a_\infty^2u_1 + a_{12}^2v_1}{a_{12}a_\infty^2} + \frac{a_{21}(u_2 + u_3^2 + u_1u_4)v_3}{a_0a_{12}u_3u_4} + \frac{a_0a_{12}(u_2 + u_1u_4)(v_2 + v_3^2 + v_1v_4)}{a_{21}a_\infty^2u_3v_3v_4} \right). \end{aligned}$$

We need to prove the following.

- (i) The non-frozen coordinates of $\mathcal{M}(\alpha_{123})$ are a'_1 and a'_2 .
- (ii) The non-frozen coordinates of $\mu_G^{\text{rot}}(\mathcal{M}(\alpha_{130}))$ are \bar{a}'_1 and \bar{a}'_2 .
- (iii) The coordinates of $h_{31} = h_{30}l^{-1}$ are (a'_0, a'_∞) .

To compute (c, d, l) we need a formula for ι . Letting $\iota_I = \iota(-, I, -)$, we have $\iota(u, k, v) = \iota_I(u, kvk^{-1})k$, so we only need a formula for ι_I (here $I \in H$ is the identity). Applying the algorithm in Section 7.3 we obtain that if $(x', y', h') = \iota_I(y, x)$, then

$$(8.20) \quad x' = \left(\frac{x_1}{\alpha_1}, \frac{x_2\alpha_1^2}{\alpha_2}, \frac{x_3\alpha_2}{\alpha_1\alpha_3}, \frac{x_4\alpha_3^2}{\alpha_2\alpha_4} \right), \quad y' = \left(\frac{y_1\alpha_4}{\alpha_5}, \frac{y_2\alpha_5\alpha_3}{\alpha_4\alpha_6}, \frac{y_3\alpha_6^2\alpha_4^2}{\alpha_5\alpha_3^2}, \frac{y_4\alpha_3}{\alpha_4\alpha_6} \right), \quad h' = \left(\frac{1}{\alpha_3}, \frac{1}{\alpha_4} \right),$$

where the α_i are given by

$$\begin{aligned}
(8.21) \quad & \alpha_1 = 1 + x_1(y_2 + y_4), \quad \alpha_2 = 1 + x_2(y_3(1 + x_1y_4)^2 + y_1(1 + x_1(y_2 + y_4))^2), \\
& \alpha_3 = 1 + x_3(y_2 + x_2y_2y_3 + y_4) + x_1(y_2 + y_4 + x_2x_3y_2y_3y_4), \\
& \alpha_4 = 1 + x_4(y_3(1 + (x_1 + x_3)y_4)^2 + y_1(1 + (x_1 + x_3)(y_2 + y_4))^2) + x_2(y_3(1 + x_1y_4)^2 + \\
& \quad y_1(1 + x_3^2x_4y_2^2y_3 + 2x_1(y_2 + y_4) + x_1^2(y_2 + y_4)^2)) \\
& \alpha_5 = 1 + x_2y_3(1 + x_1y_4)^2 + x_4y_3(1 + (x_1 + x_3)y_4)^2, \quad \alpha_6 = 1 + (x_1 + x_3)y_4.
\end{aligned}$$

Using this, together with the explicit formulas for $\Psi\Phi\Psi$, $\Psi\Phi$ and their inverses given in (8.3) and (8.4), we obtain that c , d and l are given by

$$\begin{aligned}
(8.22) \quad & l_1^{-1} = \frac{a_\infty^2 u_1 + a_{12}^2 v_1}{a_{12} a_\infty v_1} + \frac{a_{21} a_\infty (u_2 + u_3^2 + u_1 u_4) v_3}{a_0 a_{12} u_3 u_4 v_1} + \frac{a_0 a_{12} (u_2 + u_1 u_4) (v_2 + v_3^2 + v_1 v_4)}{a_{21} a_\infty u_3 v_1 v_3 v_4} \\
l_2^{-1} &= \frac{a_{21}}{a_0} + \frac{a_{21} a_\infty^2 (u_2 + u_3^2) v_4}{a_0 a_{12}^2 u_4 v_2} + \frac{2u_3(v_2 + v_1 v_4)}{v_2 v_3} + \frac{a_0 u_2}{a_{21} v_2} + \frac{a_0 a_{12}^2 u_4 (v_2^2 + v_2 v_3^2 + 2v_1 v_2 v_4 + v_1^2 v_4^2)}{a_{21} a_\infty^2 v_2 v_3^2 v_4}, \\
c_1 &= \frac{a_{12} v_1}{a_\infty l_1}, \quad c_2 = \frac{a_{21} v_2}{a_0 l_2}, \quad c_3 = u_3 + \frac{a_{21} v_3}{a_0} + \frac{a_0 a_{12}^2 u_4 (v_2 + v_3^2 + v_1 v_4)}{a_{21} a_\infty^2 v_3 v_4}, \quad c_4 = u_4 + \frac{a_{21} a_\infty^2 v_4}{a_0^2 a_{12}^2}, \\
d_1 &= \frac{a_\infty u_1}{a_{12} l_1}, \quad d_2 = \frac{a_0 u_2}{a_{21} l_2}, \quad d_3 = u_3 + \frac{a_{21} a_\infty^2 (u_2 + u_3^2) v_3}{a_0 a_{12}^2 u_4 v_1} + \frac{a_0 u_2 (v_2 + v_3^2 + v_1 v_4)}{a_{21} v_1 v_3 v_4}, \\
d_4 &= \frac{l_1^2}{l_2} \left(\frac{2u_3 v_3}{v_1} + \frac{a_{21} a_\infty^2 (u_2 + u_3^2) v_3^2}{a_0 a_{12}^2 u_4 v_1^2} + \frac{a_0 (a_\infty^2 u_2 (v_2 + v_3^2) + a_{12}^2 u_4 v_1^2 v_4)}{a_{21} a_\infty^2 v_1^2 v_4} \right).
\end{aligned}$$

By (8.7) the non-frozen coordinates of $\mathcal{M}(\alpha_{123})$ are $c_3 a_{23} = a'_1$ and $c_4 \frac{a_{32} a_\infty^2}{a_0} = a'_2$ proving (i). Also, by (8.17),

$$(8.23) \quad h_{31} = h_{30} l^{-1} = \left(\frac{a_{30}}{l_1}, \frac{a_{03}}{l_2} \right) = \left(\frac{v_1 a_{23}}{l_1 a_\infty}, \frac{v_2 a_{32}}{a_0 l_2} \right) = (a'_0, a'_\infty),$$

proving (ii). To prove (iii), let b_i denote the coordinates of $\mathcal{M}(\alpha_{130})$, we have (since $h_{13} = w_0(l)h_{30}$)

$$\begin{aligned}
(8.24) \quad & b_{01} = l_1^{-1} a_{30}, \quad b_{10} = l_2^{-1} a_{03}, \quad b_{12} = a_{30}, \quad b_{21} = a_{03}, \\
& b_{20} = a_{01}, \quad b_{02} = a_{10}, \quad b_1 = d_3 b_{12}, \quad b_2 = d_4 \frac{b_{21} b_{01}^2}{b_{10}}.
\end{aligned}$$

Plugging this into the formula (3.8) for $\mu_{C_2}^{\text{rot}}$ we obtain expressions for b'_1 and b'_2 that equal those of \bar{a}'_1 and \bar{a}'_2 in (8.18). This concludes the proof.

Remark 8.1. Given coordinates on $A_{G,S}$ as described in Section 2.3 we get a natural cocycle on each triangle of P . When s_G is trivial, these glue together to form a natural cocycle on S , hence a pair (ρ, D) . When s_G is non-trivial, the labelings of identified edges differ by s_G (see Figure 16), and we instead get a pair $(\bar{\rho}, \bar{D})$.

9. REPRESENTATIONS OF 3-MANIFOLD GROUPS

Let M be a compact, oriented 3-manifold with boundary. Recall that $\rho: \pi_1(M) \rightarrow G$ is *boundary-unipotent* if peripheral subgroups of G map to conjugates of N , and that a *decoration* of such ρ is a ρ -equivariant assignment of a coset $gN \in \mathcal{A}$ to each ideal point (boundary-component) of the universal cover of M . A *decorated representation* is a pair (ρ, D) , where ρ

is a boundary-unipotent and D is a decoration of ρ . Note that G acts on the set of decorated representations by $g(\rho, D) = (g\rho g^{-1}, gD)$. For more details on decorations, we refer to [12]. Unless otherwise stated G denotes one of the groups A_2 , B_2 , C_2 or G_2 .

9.1. Generic configurations. We shall consider a notion of genericity for configurations, which is slightly finer than that of sufficiently generic (Definition 5.1).

Definition 9.1. An element $\alpha \in \text{Conf}_3^*(\mathcal{A})$ is *generic* if the minor coordinates of α , $\text{rot}(\alpha)$, and $\text{rot}^2(\alpha)$ are all non-zero.

The set $\text{Conf}_3^{\text{gen}}$ of generic configurations in $\text{Conf}_3^*(\mathcal{A})$ is isomorphic to a Zariski open subset of T_{Q_G} . Note that μ_G^{rot} is an isomorphism (not just a birational equivalence) on this subset.

Definition 9.2. The set $\text{Conf}_4^{\text{gen}}(\mathcal{A})$ of *generic* configurations in $\text{Conf}_4^*(\mathcal{A})$ is the largest Zariski open subset U of $\text{Conf}_4^*(\mathcal{A})$ such that $\Psi_{kl}(U) \in \text{Conf}_3^{\text{gen}}(\mathcal{A}) \times_{kl} \text{Conf}_3^{\text{gen}}(\mathcal{A})$ for $kl = 02$ or 13 , and such that μ_G^{flip} defines an isomorphism from $\Psi_{02}(U)$ to $\Psi_{13}(U)$.

The formulas in Section 3.3 provide explicit defining equations for the variety $\text{Conf}_4^{\text{gen}}(\mathcal{A})$, and Proposition 6.5 provides an explicit formula for the natural cocycle.

Example 9.3. Let $G = C_2$. Given $\alpha \in \text{Conf}_4^*(\mathcal{A})$, the simplicial boundary map ε_i in (2.4) induces configurations on each of the faces with coordinates given by the map \mathcal{M} . We denote the coordinates on the i th face by $f_{1,i}$ and $f_{2,i}$, and the coordinates on the edges by a_{ij} (see Figure 21). It now follows from (3.12) that the coordinates satisfy

$$(9.1) \quad \begin{aligned} a_{20}z_1 &= a_{32}f_{1,3}^2 + f_{2,1}f_{2,3}, & a_{02}f_{1,2} &= a_{01}f_{1,1} - a_{30}f_{1,3}, & f_{1,1}z_2 &= -f_{2,1}f_{1,2} + a_{03}a_{23}f_{a,3}, \\ f_{2,1}f_{2,2} &= a_{03}z_1 + z_2^2, & f_{1,3}z_3 &= z_1 + a_{12}z_2, & f_{2,0}f_{2,3} &= a_{10}a_{12}^2a_{32} + a_{21}z_1, \\ a_{13}z_1 &= f_{2,0}f_{2,2} + a_{10}z_3^2, & a_{31}z_2 &= a_{23}f_{2,2} - f_{1,2}z_3, & f_{1,0}z_3 &= a_{12}a_{13}a_{23} + f_{2,0}a_{31} \end{aligned}$$

The ideal generated by these relations defines the Zariski closure of $\text{Conf}_4^{\text{gen}}(\mathcal{A})$ for C_2 .

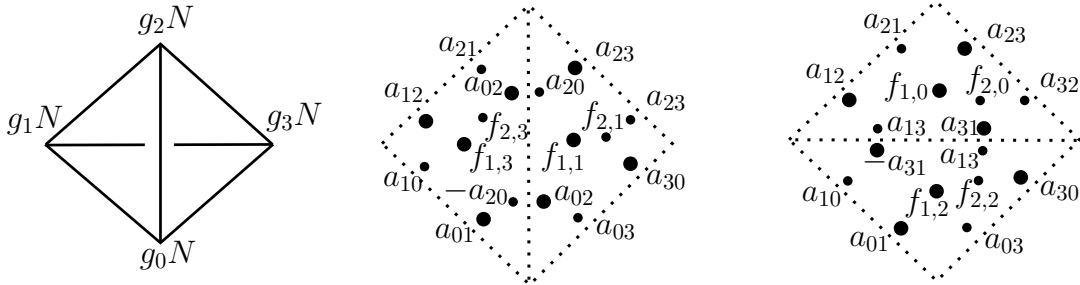


FIGURE 21. Coordinates on the faces of a simplex.

9.2. Generic decorations and the Ptolemy variety. Let \mathcal{T} be a topological ideal triangulation of M . We assume for simplicity that the triangulation is *ordered*, i.e. that we have fixed a vertex ordering of each simplex, which is respected by the face pairings. For more on ordered triangulations, see e.g. [10].

A decorated representation (ρ, D) associates a quadruple of affine flags to each 3-simplex of \mathcal{T} . We refer to the collection of such as the *associated configurations*.

Definition 9.4. A decoration of a boundary-unipotent representation ρ is *generic* if the associated configurations are in $\text{Conf}_4^{\text{gen}}(\mathcal{A})$.

Remark 9.5. Note that this notion depends on \mathcal{T} .

The triangulation \mathcal{T} defines a category J with an object for each k -simplex and a morphism for each inclusion of a k -simplex in an l -simplex. For $k = 1, 2$, let $\text{Conf}_k^{\text{gen}}(\mathcal{A}) = \text{Conf}_k^*(\mathcal{A})$.

Definition 9.6. The *Ptolemy variety* $P_G(\mathcal{T})$ is the limit of the functor from J to affine varieties taking a k -cell to $\text{Conf}_k^{\text{gen}}(\mathcal{A})$, and an inclusion onto the i th face to the face map ε_i in (2.4).

Informally, the Ptolemy variety is the variety built from copies of $\text{Conf}_4^{\text{gen}}(\mathcal{A})$ by gluing them together using the gluing pattern determined by the triangulation, i.e. if two faces are identified, the corresponding configuration spaces are identified as well. Tautologically, we have a one-to-one correspondence between points in the Ptolemy variety and generically decorated boundary-unipotent representations, i.e. (2.8) holds. The natural cocycle provides an explicit formula for this correspondence.

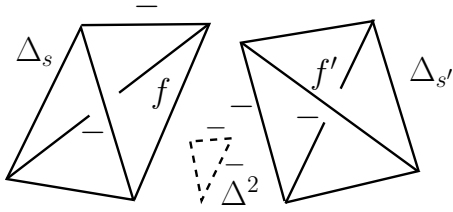
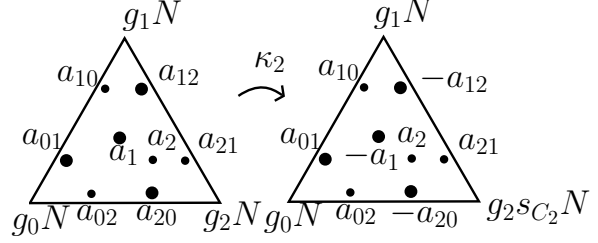
Remark 9.7. One could also consider a Ptolemy variety by gluing together copies of $\text{Conf}_4^*(\mathcal{A})$. However, we don't have explicit defining equations for this variety.

9.3. Obstruction classes. As mentioned earlier, there are interesting boundary-unipotent representations in $G/\langle s_G \rangle$ that don't have boundary-unipotent lifts to G . The obstruction is a class in $H^2(M, \partial M; \mathbb{Z}/2\mathbb{Z}) = H^2(\widehat{M}; \mathbb{Z}/2\mathbb{Z})$, where \widehat{M} is the space obtained from M by collapsing each boundary component to a point. The theory of obstruction classes developed in [12] for $\text{SL}(n, \mathbb{C})$ (see [11] for a summary when $n = 2$) has a natural analogue for G . The theory is an elementary generalization of the $\text{SL}(n, \mathbb{C})$ case, so we only sketch it.

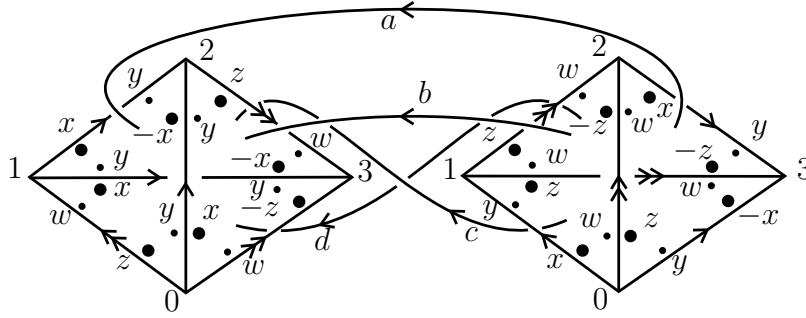
Fix an ordered triangulation \mathcal{T} . This determines a Δ -complex structure (as in Hatcher [13]) on \widehat{M} . Let $C^*(\widehat{M}; \mathbb{Z}/2\mathbb{Z})$ be the simplicial complex of $\mathbb{Z}/2\mathbb{Z}$ -valued cochains, and let $\sigma \in C^2(\widehat{M}; \mathbb{Z}/2\mathbb{Z})$ be a cocycle. The restriction σ_s of σ to a 3-simplex Δ_s of \mathcal{T} is a coboundary, so we may represent σ by a collection $\eta_s \in C^1(\Delta_s; \mathbb{Z}/2\mathbb{Z})$ such that $\delta(\eta_s) = \sigma_s$. Note that if a face f of Δ_s is identified with a face f' of $\Delta_{s'}$, then $\tau_{f,f'} = \eta_{s|f}(\eta_{s'|f'})^{-1}$ is a cocycle on Δ^2 , a standard simplex canonically identified with f and f' . Since every such is a coboundary, it is either trivial, or there exists a unique $j = j_{f,f'} \in \{0, 1, 2\}$ such that $\tau_{f,f'}$ is the coboundary of the 0-cochain on Δ^2 taking the j th vertex to $-1 \in \mathbb{Z}/2\mathbb{Z}$ (see Figure 22). We can now define the Ptolemy variety $P^\sigma(\mathcal{T})$ to be the variety obtained by gluing together copies of $\text{Conf}_4^{\text{gen}}(\mathcal{A})$ in such a way that if two faces f and f' are identified, the corresponding copies of $\text{Conf}_3^{\text{gen}}(\mathcal{A})$ are identified, not by the identity, but via the map $\kappa_j = \kappa_{j_{f,f'}}$, replacing $g_j N$ by $g_j s_G N$ (see Figure 22). Note that the effect of κ_j on the natural cocycle is to leave all three short edges and the long edge opposite j fixed, and to multiply the two long edges extending to j by s_G .

One now checks that up to a canonical isomorphism $P_G^\sigma(\mathcal{T})$ only depends on the cohomology class of σ , and that the set $Z^1(\widehat{M}; \mathbb{Z}/2\mathbb{Z})$ of 1-cocycles acts on $P_G^\sigma(\mathcal{T})$ with orbits corresponding to decorated boundary-unipotent $G/\langle s_G \rangle$ -representations. This proves (2.9).

9.4. The reduced Ptolemy variety. If M has a single boundary component, the action of H on $\text{Conf}_k^{\text{gen}}(\mathcal{A})$ where $h \in H$ acts by replacing each coset $g_i N$ by $g_i h N$ descends to an action on $P_G(\mathcal{T})$. More generally, if M has c boundary-components, we get an action by H^c . This action is also defined for $P_G^\sigma(\mathcal{T})$. We refer to the quotients as *reduced Ptolemy varieties*, and we denote the quotients by $P_G(\mathcal{T})_{\text{red}}$ and $P_G^\sigma(\mathcal{T})_{\text{red}}$, respectively.

FIGURE 22. η_s , $\eta_{s'}$ and $\eta_{s|f}(\eta_{s'|f'})^{-1}$.FIGURE 23. Effect of κ_2 on the coordinates.

9.5. Explicit computations for the figure eight not complement. Let M be the figure eight knot complement, and let \mathcal{T} be the standard ideal triangulation of M with two ideal simplices. Figure 24 shows this triangulation together with the edge coordinates for $G = C_2$ (the face coordinates are not shown).

FIGURE 24. Ordered triangulation of M .

Using the explicit relations in (9.1) we obtain that the Zariski closure of $P_{C_2}(\mathcal{T})$ is given by

$$(9.2) \quad \begin{array}{lll} f_{2,1}f_{2,3} + f_{1,3}^2w - z_{1,0}y, & f_{1,2}x - f_{1,1}z - f_{1,3}z, & z_{2,0}f_{1,1} + f_{2,1}f_{1,2} - f_{1,3}wz, \\ z_{2,0}^2 - f_{2,1}f_{2,2} + z_{1,0}w, & z_{1,0} - z_{3,0}f_{1,3} + z_{2,0}x, & f_{2,0}f_{2,3} - w^2x^2 - z_{1,0}y, \\ f_{2,0}f_{2,2} + z_{3,0}^2w - z_{1,0}y, & z_{3,0}f_{1,2} - z_{2,0}x - f_{2,2}z, & z_{3,0}f_{1,0} + f_{2,0}x - xyz, \\ f_{2,1}f_{2,3} - z_{1,1}w + f_{1,1}^2y, & xf_{1,1} + f_{1,3}x - f_{1,0}z, & z_{2,1}f_{1,3} + f_{1,0}f_{2,3} - f_{1,1}xy, \\ z_{2,1}^2 - f_{2,0}f_{2,3} + z_{1,1}y, & z_{1,1} - z_{3,1}f_{1,1} + z_{2,1}z, & f_{2,1}f_{2,2} - z_{1,1}w - y^2z^2, \\ f_{2,0}f_{2,2} - z_{1,1}w + z_{3,1}^2y, & z_{3,1}f_{1,0} - f_{2,0}x - z_{2,1}z, & z_{3,1}f_{1,2} + f_{2,2}z - wxz. \end{array}$$

A computation using Magma [3] shows that there are no solutions where all coordinates are non-zero, and where all rotations of all faces are well defined. Hence, $P_{C_2}(\mathcal{T})$ is empty.

A simple computations shows that $H^2(\widehat{M}; \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, and that the generator σ is represented by the cocycle taking the faces paired by b and c to -1 . The Zariski closure of the

Ptolemy variety $P_G^\sigma(\mathcal{T})$ is given by

$$(9.3) \quad \begin{array}{lll} f_{2,1}f_{2,3} + f_{1,3}^2w - z_{1,0}y, & f_{1,2}x - f_{1,1}z - f_{1,3}z, & z_{2,0}f_{1,1} + f_{2,1}f_{1,2} + f_{1,3}wz, \\ z_{2,0}^2 - f_{2,1}f_{2,2} + z_{1,0}w, & z_{1,0} - z_{3,0}f_{1,3} + z_{2,0}x, & f_{2,0}f_{2,3} - w^2x^2 - z_{1,0}y, \\ f_{2,0}f_{2,2} + z_{3,0}^2w - z_{1,0}y, & z_{3,0}f_{1,2} - z_{2,0}x + f_{2,2}z, & z_{3,0}f_{1,0} + f_{2,0}x + xyz, \\ f_{2,1}f_{2,3} - z_{1,1}w + f_{1,1}^2y, & xf_{1,1} + f_{1,3}x - f_{1,0}z, & z_{2,1}f_{1,3} - f_{1,0}f_{2,3} + f_{1,1}xy, \\ z_{2,1}^2 - f_{2,0}f_{2,3} + z_{1,1}y, & z_{1,1} + z_{3,1}f_{1,1} + z_{2,1}z, & f_{2,1}f_{2,2} - z_{1,1}w - y^2z^2, \\ f_{2,0}f_{2,2} - z_{1,1}w + z_{3,1}^2y, & z_{3,1}f_{1,0} + f_{2,0}x + z_{2,1}z, & z_{3,1}f_{1,2} + f_{2,2}z - wxz. \end{array}$$

One easily checks that the action by an element $(k_1, k_2) \in H$ multiplies the coordinates $f_{1,i}$ and $f_{2,i}$ by $k_1^2k_2$ and $k_1^2k_2^2$, respectively, so we may add the additional relations $f_{1,0} = 1$ and $f_{2,0} = 1$ to obtain the reduced Ptolemy variety $P_G^\sigma(\mathcal{T})_{\text{red}}$. A Magma computation shows that there are two zero-dimensional components in $P_G^\sigma(\mathcal{T})_{\text{red}}$. One is defined over $\mathbb{Q}(\sqrt{-3})$ and given by

$$(9.4) \quad \begin{aligned} f_{1,0} = f_{2,0} = f_{1,2} = -f_{2,3} = 1, \quad f_{1,1} = \frac{1}{2}(-1 + \sqrt{-3}), \quad f_{2,1} = -f_{1,2} = -f_{1,3} = \frac{1}{2}(1 + \sqrt{-3}), \\ x = \frac{1}{3}(1 + \sqrt{-3}), \quad y = \frac{3}{8}(-1 + \sqrt{-3}), \quad z = -\frac{1}{3}(1 + \sqrt{-3}), \quad w = \frac{3}{4}. \end{aligned}$$

The other component is defined over $\mathbb{Q}(\omega)$, with ω defined in (2.13), and is given by

$$(9.5) \quad \begin{aligned} f_{1,0} = f_{2,0} = 1, \quad f_{1,1} = -\frac{3\omega^5}{16} + \frac{3\omega^4}{8} - \frac{7\omega^3}{16} + \frac{7\omega^2}{8} - \frac{15\omega}{8} + \frac{3}{2}, \\ f_{2,1} = \frac{\omega^4}{2} - \frac{\omega^3}{2} + \omega^2 - 2\omega + 3, \quad f_{1,2} = -\frac{3\omega^5}{16} - \frac{\omega^4}{8} + \frac{\omega^3}{16} + \frac{3\omega^2}{8} - \frac{3\omega}{8} - \frac{3}{2}, \\ f_{2,2} = -\frac{\omega^4}{2} + \frac{\omega^3}{2} - \omega^2 + 2\omega - 3, \quad f_{1,3} = \frac{\omega^5}{16} - \frac{\omega^4}{8} + \frac{5\omega^3}{16} - \frac{\omega^2}{8} + \frac{5\omega}{8} - \frac{1}{2}, \\ x = \frac{3\omega^5}{32} - \frac{3\omega^4}{16} + \frac{7\omega^3}{32} - \frac{11\omega^2}{16} + \frac{11\omega}{16} - \frac{1}{4}, \quad y = -\frac{\omega^5}{4} + \frac{3\omega^4}{8} - \frac{5\omega^3}{8} + \frac{3\omega^2}{2} - \frac{5\omega}{4} + 1, \\ z = \frac{\omega^5}{64} + \frac{3\omega^4}{32} - \frac{3\omega^3}{64} + \frac{3\omega^2}{32} - \frac{15\omega}{32} + \frac{9}{8}, \quad w = -\frac{5\omega^5}{16} + \frac{\omega^4}{2} - \frac{11\omega^3}{16} + \frac{9\omega^2}{8} - \frac{15\omega}{8} + \frac{3}{2}. \end{aligned}$$

Remark 9.8. The reduced Ptolemy variety $P_{B_2}^\sigma(\mathcal{T})_{\text{red}}$ also has two components of degree 2 and 6 defined over $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\omega)$, respectively. This is, of course, not surprising since B_2 and C_2 are isomorphic. We have not been able to explicitly compute the Ptolemy variety for G_2 .

9.6. Recovering the representations. One can explicitly recover the representation corresponding to a point in the Ptolemy variety using the natural cocycle. As described in [22, Sec. 4.1] the fundamental group of the figure eight knot complement has a presentation of the form

$$(9.6) \quad \langle a, b, c \mid ca^{-1}bc^{-1}a, ab^{-1}c^{-1}b \rangle,$$

where a , b , and c are the face pairings in Figure 24. This presentation is isomorphic to the presentation (2.11) via the map taking x_1 to c and x_2 to ab^{-1} . Let $\alpha_{ij,s}$ and $\beta_{jk,s}^i$ denote the labelings of the natural cocycle associated to simplex s . As in [22, Sec. 3.5.1], the representation is given by

$$(9.7) \quad a = (\beta_{31,0}^2\alpha_{23,0}\beta_{12,0}^3)^{-1}, \quad b = (\beta_{01,0}^3)^{-1}(\beta_{30,1}^2)^{-1}\alpha_{23,1}, \quad c = \beta_{12,0}^3\beta_{12,1}^3.$$

The formulas differ slightly from those of [22] due to the fact that we are using an ordered triangulation. Using the Serre generators for $\mathfrak{sp}(4, \mathbb{C})$ given in Knapp [16], we obtain

$$(9.8) \quad x_1(t) = \begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -t & 1 \end{bmatrix}, \quad x_2(t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and that $h_1^t = \text{diag}(t, t^{-1}, t^{-1}, t)$ and $h_2^t = \text{diag}(1, t, 1, t^{-1})$. Also, $s_{C_2} = -I$, and $\overline{w_0} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Using this, we can now recover the natural cocycle explicitly from the coordinates, and we obtain the formulas in Section 2.4.

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UNIVERSITY OF MARYLAND, DEPARTMENT OF MATHEMATICS, COLLEGE PARK, MD 20742-4015, USA
<http://www2.math.umd.edu/~zickert>
E-mail address: `zickert@math.umd.edu`