Math 220 Exam 1 + Solution

1. (a) (18 points) By cutting away identical squares from each corner of a rectangular piece of cardboard and folding up the resulting flaps, the cardboard may be turned into an open box. If the cardboard is 16 inches long and 10 inches wide, find the dimension of the box that will yield the maximum volume.

Solution: We begin by looking a picture that illustrates the situation.

The next step is to observe that the formula for the volume $V(x)$ of the open box is

$$V(x) = (10 - 2x)(16 - 2x)x = 4x^3 - 52x^2 + 160x.$$ 

In order to find the maximum we take the first derivative and get that

$$V'(x) = 12x^2 - 104x + 160 = 4(3x^2 - 26x + 40) = 4(3x - 20)(x - 2)$$

giving us that $V'(x) = 0$ when $x = 2$ or $x = \frac{20}{3}$. However since $2 \cdot \frac{20}{3} > 10$ we are not able to cut squares that large allowing us to conclude that $x = 2$ is the cut that maximizes the volume. With $x = 2$ we get the following dimensions $2\text{inch} \times 6\text{inch} \times 12\text{inch}$.

Remark. The problem does not require you to verify that $x = 2$ truly maximizes $V(x)$. This is, however, easily done by checking that $V''(x) < 0$. 

(b) (5 points) Let \( f(t) \) be the temperature, in degrees Fahrenheit, of a cup of coffee \( t \) minutes after it has been poured. Suppose that \( f(4) = 115 \) and \( f'(4) = -5 \). What does this mean about the coffee? Estimate the temperature after 4 minutes and 12 seconds, that is, after 4.2 minutes.

**Solution:** The information provided tells us that 4 minutes after the coffee is poured the temperature of the coffee in the cup is 115°F and is decreasing at instantaneous rate of 5°F/minute. In order to approximate the temperature after 4.2 minutes we use the formula

\[
f(a + h) - f(a) \approx f'(a) \cdot h.
\]

In our case \( a = 4 \) and \( h = 0.2 \) giving us that

\[
f(4.2) - 115 \approx (-5) \cdot 0.2
\]

or equivalently

\[
f(4.2) \approx 115 - 5(0.2) = 114F.
\]

2. (a) (10 points) Determine the equation of the tangent line to the curve \( f(x) = (2x^2 - 3x)^3 \) at \( x = 1 \). Write your equation in slope intercept form.

**Solution:** We will use the point-slope formula which says that the equation for the tangent line of curve described by a function \( f(x) \) at point \( x = 1 \) is given by

\[
y - f(1) = f'(1)(x - 1).
\]

with this in mind we need to find \( f(1) \) and \( f'(1) \). We begin by

\[
f(1) = (2 \cdot 1^2 - 3 \cdot 1)^3 = (2 - 3)^3 = (-1)^3 = -1.
\]

The next step is to find the derivative \( f'(x) \)

\[
f'(x) = 3 \cdot (2x^2 - 3x)^2 (4x - 3)
\]

evaluating at \( x = 1 \) gives us

\[
f'(1) = 3(2 \cdot 1^2 - 3 \cdot 1)^2 (4 \cdot 1 - 3) = 3.
\]

Combining all of the information above we get

\[
y - (-1) = 3(x - 1)
\]

\[
y + 1 = 3x - 3
\]

\[
y = 3x - 4
\]

(b) (10 points) Let \( h(x) = \sqrt[4]{x^5} + 2\sqrt{x^3} + x \). Find \( h'(16) \).
Solution: Begin by rewriting \( h(x) \) as
\[
h(x) = 4x^{5/4} + 2x^{3/2} + x
\]
and then we take the derivative giving us
\[
h'(x) = 4 \cdot \frac{5}{4} x^{1/4} + 2 \cdot \frac{3}{2} x^{1/2} + 1 = 5\sqrt[4]{x} + 3\sqrt{x} + 1.
\]
Evaluating at \( x = 16 \) we get
\[
h'(16) = 5\sqrt[4]{16} + 3\sqrt{16} + 1 = 10 + 12 + 1 = 23.
\]
3. (a) Let \( g(x) = \frac{4x - 4}{x^2} \).
   i. (5 points) What is the domain of \( g(x) \) ?
   ii. (4 points) Find the \( x \) and \( y \) intercept. If there are none state so.
   iii. (4 points) Identify all asymptotes of \( g(x) \). Indicate if they are vertical, horizontal or slant.

Solution:

i. Since the only issue is when \( x = 0 \) in interval form the domain of the function is
\[
(-\infty, 0) \cup (0, \infty).
\]

Remark. It is important to understand that the domain is a collection of numbers and thus \( x \neq 0 \) is not correct. One could, however, also write the domain as
\[
\{ x \in \mathbb{R} \mid x \neq 0 \}
\]

\text{or}
\[
\mathbb{R}\setminus\{0\}.
\]

The first means that domain is all \( x \) which are real numbers such that \( x \neq 0 \) while the second means all real numbers except the number 0.

ii. Recall that \( x \)-intercept is when \( g(x) = 0 \) and the \( y \)-intercept is when \( x = 0 \). From part i. we know that \( x = 0 \) is not in the domain in the function and hence there is no \( y \)-intercept. For the \( x \)-intercept we have
\[
\frac{4x - 4}{x^2} = 0
\]
\[
4x - 4 = 0
\]
\[
4x = 4
\]
\[
x = 1
\]

and hence the \( x \)-intercept is the point \((1, 0)\).
iii. Since the numerator of the equation is not equal to 0 when the denominator is 0 at \( x = 0 \), \( g(x) \) has a vertical asymptote at \( x = 0 \). Next we check what happens when \( |x| \) gets very large. We have (as long as \( x \neq 0 \))

\[
g(x) = \frac{4x - 4}{x^2} = \frac{x(4 - \frac{4}{x})}{x^2} = \frac{4 - \frac{4}{x}}{x}
\]

from this it is clear that as \( |x| \) gets very large \( g(x) \) goes to 0 and hence \( y = 0 \) is a vertical asymptote and this eliminates the possibility of a slant asymptote.

(b) (14 points) Find the \( x \) coordinates of the inflection points of the function \( H(x) = \frac{1}{x}x^7 - \frac{7}{30}x^6 - x^5 \). Where is \( H(x) \) concave down? Write your answer in interval form.

**Solution:** The inflection points are when \( H''(x) = 0 \) and the intervals where the function is concave down is when \( H''(x) < 0 \). We begin by taking the first and second derivative

\[
H'(x) = x^6 - \frac{7}{5}x^5 - 5x^4
\]

and

\[
H''(x) = 6x^5 - 7x^4 - 20x^3
= x^3(6x^2 - 7x - 20)
= x^2(2x - 5)(3x + 4).
\]

this gives us that \( H''(x) = 0 \) when \( x = -4/3, 0, 5/2 \) which are also our inflection points. To analyze the concavity we do a table of signs

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\frac{4}{3})</th>
<th>0</th>
<th>( \frac{5}{2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^4 )</td>
<td>-</td>
<td>-</td>
<td>0</td>
</tr>
<tr>
<td>( 3x + 4 )</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>( 2x - 5 )</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( f''(x) )</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

Thus we see that \( H''(x) < 0 \) on \((-\infty, -4/3) \cup (0, 5/2)\).

4. (a) You are building a right-angled triangular flower garden along a stream. (the borders can be in any directions as long as they are at right angles). The fencing on the left border costs $5 per foot, while the fencing of the lower border costs $1 per foot. No fencing is required along the river. You want to spend $100 and enclose as much area as possible.

i. (2 points) Define your variables and clearly label the picture with the variables.
ii. (4 points) What is the constraint equation?

iii. (8 points) Use your constraint equation to write the objective function in terms of just one variable. You do NOT need to find the dimension of the garden.

Solution:

i. We begin with the picture of the situation

\[ x \quad y \]

where

\( x \) : length of the lower border
\( y \) : length of the left border

ii. Since our constraint is determined by that we want to spend $100 hence the constraint is

\[ x + 5y = 100 \]

iii. We recall that the area of a triangular flower garden is given by \( \frac{1}{2}xy \). From the constraint we get that

\[ x = 100 - 5y \]

combining this we get our objective function of one variable

\[ A(y) = \frac{1}{2}(100 - 5y)y = \frac{100y - 5y^2}{2} \]

(b) (16 points) Given the cost function \( C(x) = x^3 - 6x^2 + 13x + 15 \), find the minimal marginal cost.
Solution: We recall that the marginal cost is given by $C'(x)$ and so the equation for the marginal cost is given by

$$C'(x) = 3x^2 - 12x + 13.$$  

The next step is to find the critical values of $C'(x)$ which is when $C''(x) = 0$. We have

$$C''(x) = 6x - 12 = 5(x - 2)$$

hence we get that $C''(x) = 0$ when $x = 2$. This gives us that the minimal marginal cost occurs when $x = 2$ to find the minimal marginal cost we need to evaluate $C'(2)$ we get

$$C'(2) = 3 \cdot 2^2 - 12 \cdot 2 + 13 = 1.$$  

Remark. Just like in problem 1. one might want to verify that that the it is in fact in minimum at $x = 2$ to do this one would have to check that $C'''(3) = 6 > 0$.  

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