Expository Notes on *Introduction to Harmonic Analysis*

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The following is a continuing study of Yitzhak Katznelson’s *An Introduction to Harmonic Analysis* [Katz]. I present some of the results with Katznelson’s proof often with expanded details. In some cases, I even provide alternate proofs to some of the results.

1 Recap and the Schwartz Space

See previous document for material from sections 1-3 of Chapter 6 of [Katz]. We recall some of the important results here:

**Definition 1.1.** The Fourier Transform, \( \hat{f} \), of \( f \) is defined by

\[
\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i \xi x} \, dx
\]

for all real-valued \( \xi \).

(Note: we deviate from the definition of \( \hat{f} \) in [Katz] and use the more conventional definition above). Notice that \( f(x) \) is defined for all \( x \in \mathbb{R} \) and \( \hat{f}(\xi) \) is defined for all \( \xi \in \mathbb{R} \). Therefore it is helpful to distinguish these two domains. We say \( x \in \mathbb{R} \) and \( \xi \in \mathbb{R} \).

**Theorem 1.2.** For \( f, g \in L^1(\mathbb{R}) \) then

1. \( (\hat{f} + \hat{g})(\xi) = \hat{f}(\xi) + \hat{g}(\xi) \).
2. For any \( \alpha \in \mathbb{C} \), \( (\alpha \hat{f})(\xi) = \alpha \hat{f}(\xi) \).
3. Let \( \hat{f} \) denote the complex conjugate of \( f \). Then \( \hat{\hat{f}}(\xi) = \hat{f}(\xi) \).
4. For \( \lambda \in \mathbb{R} \setminus \{0\} \), denote \( f_\lambda(x) = \lambda f(\lambda x) \). Then \( \hat{f}_\lambda(\xi) = \frac{1}{\lambda} \hat{f}(\frac{\xi}{\lambda}) \).
5. For \( y \in \mathbb{R} \) let \( (\tau_y f)(t) = f(t - y) \). Then \( \hat{\tau_y f}(\xi) = \hat{f}(\xi)e^{-2\pi i \xi y} \).
6. \(|\hat{f}(\xi)| \leq \|f\|_{L^1(\mathbb{R})} \).

**Definition 1.3.** The Schwartz space, \( \mathcal{S}(\mathbb{R}) \), is the space of \( C^\infty \) functions that are rapidly decreasing, in the sense that

\[
\sup_{x \in \mathbb{R}} |x|^k |f^{(l)}(x)| < \infty, \quad \text{for integers } k, l \geq 0.
\]
Theorem 1.4. If \( f \in \mathcal{S}(\mathbb{R}) \) then,

1. \( \hat{f}'(\xi) = 2\pi i \xi \hat{f}(\xi) \).
2. \( (-2\pi i x f(x)) = \frac{d}{dx} \hat{f}(\xi) \).

Theorem 1.5. If \( f \in \mathcal{S}(\mathbb{R}) \), then \( \hat{f} \in \mathcal{S}(\hat{\mathbb{R}}) \).

Proof. Since the Fourier transform is the integral of the product of two smooth functions, we can say that \( \hat{f} \in C^\infty(\hat{\mathbb{R}}) \). Since \( f \in \mathcal{S}(\mathbb{R}) \) decays faster than any polynomial, it is integrable. Thus by Theorem 1.2 (property 6), \( \hat{f}(\xi) \) is bounded for all \( \xi \in \hat{\mathbb{R}} \). This implies that the expression \( |\xi|^k \frac{d^k}{dx^k} \hat{f}(\xi) \) must also be bounded since (by the previous theorem) this is precisely the Fourier transform of \( \frac{1}{(2\pi)^k} \frac{d^k}{dx^k} [(-2\pi i x)^l f(x)] \). This last expression must also be in the Schwartz space since multiplication of \( f \in \mathcal{S}(\mathbb{R}) \) by polynomials preserves the rapid decay.

We discuss further properties of the Schwartz space without proofs.

We saw that the Fourier transform maps \( \mathcal{S}(\mathbb{R}) \) to \( \mathcal{S}(\hat{\mathbb{R}}) \). Moreover, this transformation is injective and continuous. Then by the Fourier inversion formula, we get that the inverse Fourier transform is also injective and continuous (i.e. \( \mathcal{F} \) is bijective and bicontinuous from \( \mathcal{S}(\mathbb{R}) \) to \( \mathcal{S}(\mathbb{R}) \)).

The Schwartz space \( \mathcal{S}(\mathbb{R}) \) is a topological vector space. We are equipped with the seminorm for \( f \in \mathcal{S}(\mathbb{R}) \),

\[ |f|_{j,n} = \sup_{x \in \mathbb{R}} |x^n f^{(j)}(x)|. \]

This seminorm induces a metric \( \rho(f,g) = \sum_{j,n \geq 0} \frac{1}{2^{j+n}} \left| \frac{1}{1 + |f - g|_{j,n}} \right|^2 \).

Under this metric, \( \mathcal{S}(\mathbb{R}) \) is complete.

2 Tempered Distributions and Pseudo-measures

When studying the Fourier transform operator, \( \mathcal{F} \), in \( L^p \) spaces, we saw that the Hausdorff-Young Theorem defines \( \mathcal{F} \) for \( 1 < p \leq 2 \) but fails when \( p > 2 \). Moreover, there is no homogeneous Banach space \( \mathcal{B} \) on \( \mathbb{R} \) such that for \( p > 2 \) we have \( \|\mathcal{F} f\|_\mathcal{B} \leq C \|f\|_{L^p(\mathbb{R})} \) for all \( f \in L^1 \cap L^\infty(\mathbb{R}) \). So a new approach is necessary to define the Fourier transform on these spaces. We would, ideally, like to preserve Parseval’s formula and the inversion formula.

Definition 2.1. A tempered distribution, \( \mu \), on \( \mathbb{R} \) is a continuous linear functional on \( \mathcal{S}(\mathbb{R}) \). That is, \( \mu \) is an element of the dual of the Schwartz space, \( \mathcal{S}^*(\mathbb{R}) \).

Given a tempered distribution \( \mu \in \mathcal{S}^*(\mathbb{R}) \), we can define \( \hat{\mu} \) as the tempered distribution on \( \hat{\mathbb{R}} \) satisfying the relation:

\[ \langle \hat{f}, \hat{\mu} \rangle = \langle f, \mu \rangle. \]

The space \( \mathcal{S}^*(\mathbb{R}) \) is quite large. For example, take any \( g \) which is measurable, locally integrable, and bounded above by a polynomial of \( x \) near infinity. Then \( g \) can be identified with a tempered distribution \( \mu \) given by \( \langle f, \mu \rangle = \int f(x)g(x) \, dx \) for all \( f \in \mathcal{S}(\mathbb{R}) \). The same is true for any \( g \in L^p(\mathbb{R}) \) for \( p \geq 1 \). Similarly, if
ν ∈ M₀(ℝ), then ⟨f, µ⟩ = ∫ f(x) dν(x) also defines a tempered distribution in $S^*(ℝ)$. The fact that $S^*(ℝ)$ is large proves to be a disadvantage when trying to examine the Fourier transform of a tempered distribution. The question we want to examine is given a µ ∈ $S^*(ℝ)$, what can we say about µ.

Definition 2.2. We say that a distribution ν ∈ $S^*(ℝ)$ vanishes on the open set $O ⊆ ℝ$ provided $⟨φ, ν⟩ = 0$ for every $φ ∈ S(ℝ)$ with compact support contained in $O$.

Lemma 2.3. Let $O_1, O_2 ⊆ ℝ$ be both open and let $K ⊆ O_1 ∪ O_2$ be compact. Then there exist two functions $φ_1, φ_2 ∈ C^∞₀(ℝ)$ satisfying $\text{supp}(φ_1) ⊆ O_1$, $\text{supp}(φ_2) ⊆ O_2$, and $φ_1 + φ_2 ≡ 1$ everywhere on $K$.

Proof. This is a standard technique in PDE. The key is to utilize mollifiers. First, select two open sets, $U_1, U_2$, such that both have compact closure and are included in $O_1$ and $O_2$, respectively. Pick $ε > 0$ such that $ε$ is less than the following three values: $\text{dist}(K, ∂(U_1 ∪ U_2)), \text{dist}(U_1, ℝ \setminus O_1)$, and $\text{dist}(U_2, ℝ \setminus O_2)$ (where $\text{dist}(A, B) := \inf\{|a − b| : a ∈ A, b ∈ B\})$. Let $η ∈ C^∞(ℝ)$ such that $∥η∥₃ = 1$ and $\text{supp}(η) ⊆ (-ε, ε)$. Take $ψ_1 = 1_{U_1}$ and $ψ_2 = 1_{U_2 \setminus U_1}$. Then one can verify that $φ_1 = ψ_1 * η$ and $φ_2 = ψ_2 * η$ satisfy the conditions of the lemma.

Corollary 2.4. If ν ∈ $S^*(ℝ)$ vanishes on both $O_1$ and $O_2$, then it vanishes on $O_1 ∪ O_2$.

Proof. Let $f ∈ S(ℝ)$ be such that $\text{supp}(f) = K ⊆ O_1 ∪ O_2$ is compact. Let $φ_1$ and $φ_2$ be the functions prescribed in the previous lemma, that is, $\text{supp}(φ_1) ⊆ O_1$, $\text{supp}(φ_2) ⊆ O_2$, and $φ_1 + φ_2 ≡ 1$ everywhere on $K$. We can easily verify that $φ_1, φ_2 ∈ S(ℝ)$ since each is $C^∞$ and compact support provides the necessary “rapid decay”. Then for any $x ∈ ℝ$ we have

$$f(x) = f(x)(φ_1(x) + φ_2(x)) = f(x)φ_1(x) + f(x)φ_2(x)$$

since $φ_1 + φ_2 ≡ 1$ on $K$. Notice that $f(x)φ_1(x)$ (respectfully $f(x)φ_2(x)$) has compact support in $O_1$ (respectfully $O_2$). This implies that $⟨fφ_1, ν⟩ = 0$ and $⟨fφ_2, ν⟩ = 0$ by our hypothesis. Since $ν ∈ S^*(ℝ)$ is linear, we have

$$0 = ⟨fφ_1, ν⟩ + ⟨fφ_2, ν⟩ = ⟨fφ_1 + fφ_2, ν⟩ = ⟨f, ν⟩.$$

Since $f$ was arbitrary with $\text{supp}(f) = K ⊆ O_1 ∪ O_2$, we are done.

This corollary can easily be extended to finite unions of open sets on which ν vanishes. This can then be extended to arbitrary unions since we impose that our test functions have compact support in ℝ.

Definition 2.5. The support of ν ∈ $S^*(ℝ)$, denoted supp(ν), is the complement of the largest open set $O ⊆ ℝ$ on which ν vanishes.

Notice that if ν is identified with a continuous function $g$, then $\text{supp}(ν) = \text{supp}(g)$. Another property is that if φ ∈ $S(ℝ)$ has compact support disjoint from supp(ν) then necessarily $⟨φ, ν⟩ = 0$. In addition, it is simple to show that if supp(ν) = then ν ≡ 0.
Theorem 2.7. A distribution \( \hat{\nu} \in \mathcal{S}^*(\mathbb{R}) \) is continuous on \( \mathcal{S}(\mathbb{R}) \) with respect to the norm induced by \( \mathcal{FL}^p \) if and only if \( \hat{\nu} \in \mathcal{FL}^q \) for \( \frac{1}{p} + \frac{1}{q} = 1 \).

Proof. (\( \Leftarrow \)) Let \( \hat{\nu} \in \mathcal{FL}^q \), that is, \( \hat{\nu} \) is the Fourier transform of some \( g \in \mathcal{L}^q(\mathbb{R}) \). Hence for any \( \hat{f} \in \mathcal{S}(\mathbb{R}) \)

\[
|\langle \hat{f}, \hat{\nu} \rangle| = |\langle \hat{f}, \hat{g} \rangle| = |\langle f, g \rangle| \leq \|f\|_{\mathcal{L}^p(\mathbb{R})} \|g\|_{\mathcal{L}^q(\mathbb{R})}
\]

\[
= \left\| \hat{f} \right\|_{\mathcal{FL}^p(\mathbb{R})} \|\hat{g}\|_{\mathcal{L}^q(\mathbb{R})} = \left\| \hat{f} \right\|_{\mathcal{FL}^p(\mathbb{R})} \|\hat{\nu}\|_{\mathcal{L}^q(\mathbb{R})}
\]

Hence, \( \hat{\nu} \) is a bounded, therefore continuous, linear operator (with respect to the \( \mathcal{FL}^p \) norm).

(\( \Rightarrow \)) Now suppose \( \hat{\nu} \in \mathcal{S}^*(\mathbb{R}) \) is continuous with respect to the \( \mathcal{FL}^p \) norm. Then for any \( \hat{f} \in \mathcal{FL}^p \) we have \( \langle \hat{f}, \hat{\nu} \rangle = \langle f, \nu \rangle \). Then by the Riesz Representation Theorem, there exists a unique \( g \in \mathcal{L}^q(\mathbb{R}) \) such that \( \langle f, \nu \rangle = \int f g \) for all \( f \in \mathcal{L}^p(\mathbb{R}) = \langle f, g \rangle \). So we now have

\[
\langle \hat{f}, \hat{\nu} \rangle = \langle f, \nu \rangle = \langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle.
\]

Hence \( \hat{\nu} \) is identified with \( \hat{g} \) and therefore we conclude that \( \hat{\nu} \in \mathcal{FL}^q \).

We turn now to the setting of \( \mathcal{FL}^\infty \). Suppose that \( \mu \in M_0(\mathbb{R}) \), then \( \mu \) is the Fourier transform of the \( L^\infty \) function \( h(x) = \int_{\mathbb{R}} e^{2\pi i \xi x} d\mu(\xi) \). Therefore we have \( M_0(\mathbb{R}) \subset \mathcal{FL}^\infty \). We have that \( M_0(\mathbb{R}) \) is a proper subset of \( \mathcal{FL}^\infty \) since in general, if \( \varphi \in L^\infty(\mathbb{R}) \) is not uniformly continuous then \( \hat{\varphi} \) is not a measure. But we call elements in \( \mathcal{FL}^\infty \) pseudo-measures.

Definition 2.8. The convolution of pseudo-measures \( \hat{h}_1 \) and \( \hat{h}_2 \), denoted \( \hat{h}_1 \ast \hat{h}_2 \) is the Fourier transform of \( h_1 h_2 \) (Recall that by definition of pseudo-measure, both \( h_1, h_2 \in \mathcal{L}^\infty(\mathbb{R}) \)).

We will need the following fact below to prove the subsequent lemma.

Lemma 2.9. The Schwartz space, \( \mathcal{S}(\mathbb{R}) \), is dense in \( \mathcal{L}^\infty(\mathbb{R}) = \mathcal{L}^1(\mathbb{R})^* \) in the weak-star topology. That is, for any \( h \in \mathcal{L}^\infty(\mathbb{R}) \), there exists a sequence of functions \( \{h_n\}_{n=1}^\infty \subset \mathcal{S}(\mathbb{R}) \) such that \( \lim_{n \to \infty} \langle f, h_n \rangle = \langle f, h \rangle \) for every \( f \in \mathcal{L}^1(\mathbb{R}) \).
**Proof.** Fix \( h \in L^\infty(\mathbb{R}) \) and \( \epsilon > 0 \). Choose our sequence \( \{h_n\}_{n=1}^\infty \) of smooth Schwartz functions such that each \( h_n \) has the following properties: \( \text{supp}(h_n) \subseteq (-n-1, n+1) \); \( h_n = h \) on \([-n, n] \); and \( |h_n| \leq \|h\|_{L^\infty(\mathbb{R})} \). Then we can compute

\[
\left| \int f \hat{h} - \int \hat{f} h_n \right| = \left| \int f \left( \hat{h} - \int f h_n - \int_{-n}^{-n-1} f h_n - \int_{n}^{n+1} f h_n \right) \right| \\
= \left| \int_{|x| \geq n} f \hat{h} - \int_{-n}^{-n-1} f h_n - \int_{n}^{n+1} f h_n \right| \\
\leq \int_{|x| > n} |f h| + \int_{-n}^{-n-1} |f h_n| + \int_{n}^{n+1} |f h_n| \\
\leq 3 \|h\|_{L^\infty(\mathbb{R})} \int_{|x| > n} |f|.
\]

The right-hand side vanishes for every \( f \in L^1(\mathbb{R}) \) as \( n \to \infty \) and the lemma is proved. \( \square \)

**Lemma 2.10.** Let \( h_1 \in L^\infty(\mathbb{R}) \) and \( h_2 \in L^1 \cap L^\infty(\mathbb{R}) \), then

\[
(\hat{h}_1 \ast \hat{h}_2)(\xi) = \langle \hat{h}_2(\xi - \eta), \hat{h}_1(\eta) \rangle.
\]

**Proof.** Since \( h_1 \in L^\infty \) and \( h_2 \in L^1 \cap L^\infty(\mathbb{R}) \), this gives \( h_1 h_2 \in L^1 \cap L^\infty(\mathbb{R}) \). Therefore, the Fourier transform of the product exists for every \( \xi \in \mathbb{R} \) and \( \hat{h}_1 \hat{h}_2 = \hat{h}_1 \ast \hat{h}_2 \in A(\mathbb{R}) \).

Suppose first that \( h_1 \in S(\mathbb{R}) \). Then we have

\[
(\hat{h}_1 \ast \hat{h}_2)(\xi) = \int_{\mathbb{R}} h_1(x) h_2(x) e^{-2\pi i \xi x} \, dx \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{h}_1(\eta) h_2(x) e^{-2\pi i (\xi - \eta) x} \, d\eta \, dx \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} \hat{h}_1(\eta) h_2(x) e^{-2\pi i (\xi - \eta) x} \, dx \, d\eta \\
= \int_{\mathbb{R}} \hat{h}_2(\xi - \eta) \hat{h}_1(\eta) \, d\eta = \langle \hat{h}_2(\xi - \eta), \hat{h}_1(\eta) \rangle.
\]

where the switching of the order of the integrals is justified by Fubini’s Theorem. That is, both orientations integrate to finite numbers since \( h_1 \in S(\mathbb{R}) \), hence \( \hat{h}_1 \in \mathcal{S}(\mathbb{R}) \) and Schwartz functions are in \( L^p \) for \( p \geq 1 \).

Finally, by the previous lemma, we use the fact that \( \mathcal{S}(\mathbb{R}) \) is dense in \( L^\infty(\mathbb{R}) \) in the weak-star topology. Then the fact that both sides of (2.1) depend on \( h_1 \) continuously with respect to the weak-star topology, the lemma holds for any \( h_1 \in L^\infty(\mathbb{R}) \). \( \square \)

We now show that a pseudo-measure with a finite support is a measure. We first show the following specific case:

**Theorem 2.11.** A pseudo-measure carried by one point is a measure.

**Proof.** Let \( h \in L^\infty(\mathbb{R}) \) and \( \text{supp}(\hat{h}) = \{0\} \) (without loss of generality, we assume the one point is 0. In the general case, multiply by \( e^{2\pi i u} \)).
If \( \varphi_1, \varphi_2 \in A(\mathbb{R}) \) and \( \varphi_1(\xi) = \varphi_2(\xi) \) in some neighborhood of \( \xi = 0 \) then 
\( \langle \varphi_1, \hat{h} \rangle = \langle \varphi_2, \hat{h} \rangle \).

Let \( c = \langle \varphi, \hat{h} \rangle \) where \( \varphi \) is any function in \( A(\mathbb{R}) \) with \( \varphi(\xi) \equiv 1 \) in some neighborhood of \( \xi = 0 \). As usual, we denote the F¥jer kernel by \( K \). Recall that 
\( \hat{K}(\xi) = \max\{1 - |\xi|, 0\} \). Then by the previous lemma, we have

\[
(2.2) \quad \hat{h}K(\xi) = \langle \hat{K}(\xi - \eta), \hat{h}(\eta) \rangle.
\]

First consider \( |\xi| \geq 1 \) then \( \text{supp} (\hat{K}) \cap \text{supp}(\hat{h}) = \emptyset \). And so \( \hat{h}K(\xi) = 0 \). Second, consider \(-1 < \xi_1 < \xi_2 < 0 \). In this case, one can calculate that
\( \hat{K}(\xi_2 - \eta) - \hat{K}(\xi_1 - \eta) = \xi_2 - \xi_1 \) for \( \eta \) sufficiently small (in particular, for \( \eta < \min\{\xi_1 + 1, -\xi_2\} \)). Then by (2.2) we have
\[
\hat{h}K(\xi_2) - \hat{h}K(\xi_1) = \langle \hat{K}(\xi_2 - \eta), \hat{h}(\eta) \rangle - \langle \hat{K}(\xi_1 - \eta), \hat{h}(\eta) \rangle
\]
\[
= \langle \hat{K}(\xi_2 - \eta) - \hat{K}(\xi_1 - \eta), \hat{h}(\eta) \rangle
\]
\[
= \langle \xi_2 - \xi_1, \hat{h}(\eta) \rangle = (\xi_2 - \xi_1)\langle 1, \hat{h}(\eta) \rangle = c(\xi_2 - \xi_1)
\]
where the last equality follows by our definition of \( c \). This says that \( \hat{h}K(\xi_2) - \hat{h}K(\xi_1) = c(\xi_2 - \xi_1) \), hence \( \hat{h}K \) is continuous in \( \xi \). Sending \( \xi_1 \to 0 \) gives
\[
\hat{h}K(\xi) = c(1 + \xi) = c\hat{K}(\xi) \quad \text{for} \quad -1 \leq \xi \leq 0.
\]
Lastly, consider \( 0 < \xi_1 < \xi_2 < 1 \). Then, again, for \( \eta \) sufficiently small, we have
\[
\hat{K}(\xi_1 - \eta) - \hat{K}(\xi_2 - \eta) = (1 - \xi_1) - (1 - \xi_2) = \xi_2 - \xi_1.
\]
This gives \( \hat{h}K(\xi_1) - \hat{h}K(\xi_2) = c(\xi_2 - \xi_1) \). Hence \( \hat{h}K(\xi) \) is continuous in this range and sending \( \xi_2 \to 1 \) gives
\[
\hat{h}K(\xi) = c(1 - \xi) = c\hat{K}(\xi) \quad \text{for} \quad 0 \leq \xi \leq 1.
\]

By combining all three cases considered above, we can conclude that \( \hat{h}K(\xi) = c\hat{K}(\xi) \) for all \( \xi \in \mathbb{R} \). Then by the uniqueness theorem, \( h(x)K(x) = cK(x) \) a.e. in \( \mathbb{R} \) and hence \( h(x) = c \) a.e. in \( \mathbb{R} \). This implies that \( \hat{h} = c\delta(0) \), which is a measure.

For general pseudo-measures supported on any finite set, simply repeat the procedure outlined in the above proof for each point in the support (up to a dilation of the \( K \) to ensure that the support of each \( \hat{h}K(\xi) \) is mutually disjoint).

**Exercise 2.12** (Katznelson VI.4.3). What is a function \( h \in L^\infty(\mathbb{R}) \) such that \( \text{supp}(\hat{h}) \) is finite?

**Solution.** Consider \( g(\xi) = \delta_0(\xi) \in \mathcal{F}L^\infty \). Then by the inversion formula \( h(x) = g(x)^\vee = \int_\mathbb{R} e^{2\pi i \xi x} \delta_0(\xi) \, d\xi = e^0 = 1 \).

In general suppose \( g(\xi) = \sum_{i=1}^N a_i \delta_{\xi_i}(\xi) \) where each \( a_i \in \mathbb{R} \) and \( \xi_i \in \mathbb{R} \). Then \( h(x) = g(x)^\vee = \int_\mathbb{R} e^{2\pi i \xi x} \sum_{i=1}^N a_i \delta_{\xi_i}(\xi) \, d\xi = \sum_{i=1}^N a_i e^{2\pi i \xi x} \).

**Remark 2.13.** Some remarks on the spaces \( \mathcal{F}L^p \) with \( 2 < p < \infty \). There is no inclusion relation between \( \mathcal{F}L^p \) and \( \mathcal{F}L^{p'} \) globally. However, locally \( \mathcal{F}L^p \subseteq \mathcal{F}L^{p'} \) for \( p < p' \). In particular, this means that all distributions are locally pseudo-measures.

There are other conclusions we can make if the support of a distribution is compact which we present in the following theorem:
Theorem 2.14. If $\nu \in \mathcal{F}L^p$ and $\text{supp}(\nu)$ is compact, then $\nu \in \mathcal{F}L^\infty$.

Proof. Say that $\text{supp}(\nu) \subseteq (-\lambda, \lambda)$. We have $\nu \in \mathcal{F}L^p$ and now consider $\mu = V_\lambda \nu$ where $V_\lambda$ is the de la Vallée Poussin kernel. (Recall that $\hat{V}_\lambda(\xi)$ vanishes on $|\xi| > 2\lambda$, is identically 1 on $(-\lambda, \lambda)$, and decays linearly in between. Thus we have $\mu = \nu$ on $(-\lambda, \lambda)$. Therefore, $\text{supp}(\mu - \nu) \cap (-\lambda, \lambda) = \emptyset$. Therefore, if $\nu = \hat{f}$ for $f \in L^p(\mathbb{R})$ then $\mu = \hat{V}_\lambda \hat{f} = \hat{V}_\lambda \ast \hat{f}$. Finally $\hat{V}_\lambda \ast f \in L^p \cap L^\infty(\mathbb{R})$. Hence $\mu = \nu \in \mathcal{F}L^\infty$. \qed

Exercise 2.15 (Katznelson VI.4.9). Fourier transforms of functions in $C_0(\mathbb{R})$ (continuous functions that vanish as $|x| \to \infty$) are called pseudo-functions on $\mathbb{R}$. Show that if $f \in L^1(\mathbb{R})$, then $f$ is a pseudo-function.

Solution. We wish to use the inversion formula to say that

$$f(x) = \int_\mathbb{R} \hat{f}(\xi) e^{2\pi i \xi x} d\xi \in C_0(\mathbb{R}).$$

We first show that $\hat{f}'(x)$ is continuous. Consider $\lim_{\delta \to 0} \int_\mathbb{R} \hat{f}'(\xi) e^{2\pi i \xi x} \, d\xi = \lim_{\delta \to 0} \int_\mathbb{R} \hat{f}(\xi) (e^{2\pi i \xi (x+\delta)} - e^{2\pi i \xi (x-\delta)}) \, d\xi$. We can assert that the integrand in absolute value is bounded above by $2f \in L^1(\mathbb{R})$. Thus by the Lebesgue dominated convergence theorem, $\lim_{\delta \to 0} \int_\mathbb{R} \hat{f}'(\xi) e^{2\pi i \xi x} \, d\xi = \lim_{\delta \to 0} \int_\mathbb{R} \hat{f}(\xi) (e^{2\pi i \xi (x+\delta)} - e^{2\pi i \xi (x-\delta)}) \, d\xi = 0$. So $\hat{f}'(x)$ is indeed continuous.

To show the function vanishes, consider

$$\lim_{x \to -\infty} \hat{f}'(x) = \lim_{x \to -\infty} \int_\mathbb{R} \hat{f}(\xi) e^{2\pi i \xi x} \, d\xi = \int_\mathbb{R} -\hat{f}(-\xi) e^{-2\pi i \xi x} \, d\xi = \int_\mathbb{R} -\hat{f}(\xi) e^{-2\pi i \xi x} \, d\xi = -\hat{g}(x)$$

where $g(\xi) = f(-\xi)$. Clearly $g(\xi) \in \hat{\mathbb{R}}$ and so by the Riemann-Lebesgue Lemma, $\lim_{x \to -\infty} \hat{g}(x) = 0$. A similar calculation for $x \to \infty$ and we’re done. \qed

3 Almost-periodic functions on the line

Definition 3.1. Consider $f : \mathbb{R} \to \mathbb{C}$ and $\epsilon > 0$. An $\epsilon$-almost-period of $f$ is a number $\tau$ such that

$$\sup_{x \in \mathbb{R}} |f(x - \tau) - f(x)| < \epsilon.$$  

For any periodic function with period $\tau$, then any integer multiple of $\tau$ is an $\epsilon$-almost-period. For any uniformly continuous function, the $\delta > 0$ that responds to the $\epsilon$ challenge is an $\epsilon$-almost-period. Notice that any function has the trivial almost-period $\tau = 0$.

Definition 3.2. The function $f : \mathbb{R} \to \mathbb{C}$ is (uniformly) almost-periodic on $\mathbb{R}$ provided it is continuous and for all $\epsilon > 0$ there exists some number (dependent on $\epsilon$ and $f$), $\Lambda = \Lambda(\epsilon, f)$, such that every interval of length $\Lambda$ contains an $\epsilon$-almost-period, $\tau$, of $f$. We denote the set of all almost-periodic functions by $AP(\mathbb{R})$.

An example of almost periodic functions include continuous periodic functions.
Lemma 3.3. Almost-periodic functions are bounded.

Proof. Let \( f \) be almost-periodic and fix \( \epsilon > 0 \). Take \( \Lambda = \Lambda(\epsilon, f) \). Pick an arbitrary \( x \in \mathbb{R} \). Then in the interval \([x - \Lambda, x]\) there exists an \( \epsilon \)-almost-period \( \tau \in [x - \Lambda, x] \). That gives

\[
\|f(x) - f(x - \tau)\| \leq |f(x) - f(x - \tau)| < \epsilon,
\]

where the left-most inequality is due to the triangle inequality. Hence

\[
-\epsilon + |f(x - \tau)| < |f(x)| < |f(x - \tau)| + \epsilon.
\]

We don’t know exactly what value \( \tau \in [x - \Lambda, x] \) is, but we do know that \( 0 \leq x - \tau \leq \Lambda \) holds regardless of the choice of \( x \) made. Hence we can conclude that

\[
-\epsilon + \inf_{0 \leq y \leq \Lambda} |f(y)| < |f(x)| < \sup_{0 \leq y \leq \Lambda} |f(y)| + \epsilon
\]

holds for all \( x \in \mathbb{R} \).

Corollary 3.4. If \( f \in AP(\mathbb{R}) \) then \( f^2 \in AP(\mathbb{R}) \).

Proof. Without loss of generality, we may assume that \( |f(x)| \leq 1/2 \). If not, scalar multiplication will give the appropriate bound while preserving almost-periodicity.

Notice that we can factor the following expression:

\[
f(x - \tau)^2 - f(x)^2 = (f(x - \tau) + f(x))(f(x - \tau) - f(x))
\]

so then if \( \tau \) is an \( \epsilon \)-almost-period of \( f \), we have

\[
|f(x - \tau)^2 - f(x)^2| = |f(x - \tau) + f(x)||f(x - \tau) - f(x)| \leq |f(x - \tau) - f(x)| < \epsilon.
\]

Hence \( \tau \) is also an \( \epsilon \)-almost-period of \( f^2 \).

Lemma 3.5. Almost-periodic functions are uniformly continuous.

Proof. Fix \( \epsilon > 0 \) and let \( \Lambda = \Lambda(\epsilon/3, f) \). We know that \( f \) is continuous on \( \mathbb{R} \), hence \( f \) is uniformly continuous on \([0, \Lambda]\). Therefore, there exists some number \( \delta_0 > 0 \) such that for all \( |\delta| < \delta_0 \),

\[
\sup_{0 \leq x \leq \Lambda} |f(x + \delta) - f(x)| < \epsilon/3.
\]

Now fix an arbitrary \( y \in \mathbb{R} \). Let \( \tau \) be an \( \epsilon/3 \)-almost-period for \( f \) contained in the interval \([y - \Lambda, y]\). Then we can write

\[
f(y + \delta) - f(y) = (f(y + \delta) - f(y + \delta - \tau)) + (f(y - \tau + \delta) - f(y - \delta)) + (f(y - \tau) - f(y)).
\]

The first and third term on the right-hand side of the above equation are each bounded by \( \epsilon/3 \) since \( \tau \) is an \( \epsilon/3 \)-almost-period of \( f \). The second term is also bounded by \( \epsilon/3 \) by the uniform continuity condition and the fact that \( 0 \leq y - \tau \leq \Lambda \) and \( |\delta| < \delta_0 \).

Definition 3.6. For any function \( f \in L^\infty(\mathbb{R}) \) we denote by \( W_0(f) \) to be the set of all translates of \( f \). That is \( W_0(f) = \{T_yf\}_{y \in \mathbb{R}} \) where \( T_yf = f(x - y) \).
Theorem 3.7. A function \( f \in L^\infty(\mathbb{R}) \) is almost-periodic iff \( W_0(f) \) is precompact in norm topology of \( L^\infty(\mathbb{R}) \).

Recall that a set in a complete metric space is precompact (that is, has compact closure) if and only if it is totally bounded (that is, for every \( \epsilon > 0 \) the set can be covered with a finite number of balls with radius \( \epsilon \)).

Proof. (\( \Rightarrow \)) Suppose \( f \) is almost-periodic. Fix \( \epsilon > 0 \) and let \( \Lambda = \Lambda(\epsilon/2, f) \). Since \( f \) is uniformly continuous, there exist a finite number of positive values \( \{\delta_1, ..., \delta_M\} \subseteq [0, \Lambda] \) such that for every \( y_0 \in [0, \Lambda] \),

\[
\inf_{1 \leq j \leq M} \left\| T_{y_0} f - T_{\delta_j} f \right\|_{L^\infty(\mathbb{R})} < \epsilon/2.
\]

Now pick some arbitrary \( y \in \mathbb{R} \) and let \( \tau \in [y - \Lambda, y] \) be an \( \epsilon/2 \)-almost-period of \( f \). By letting \( y_0 = y - \tau \) (hence \( y_0 \in [0, \Lambda] \)), we have \( \| T_y f - T_{y_0} f \|_{L^\infty(\mathbb{R})} < \epsilon/2 \) by the almost-periodicity of \( f \). Therefore, by the triangle inequality, we can conclude that

\[
\inf_{1 \leq j \leq M} \left\| T_y f - T_{\delta_j} f \right\|_{L^\infty(\mathbb{R})} < \epsilon.
\]

But \( y \in \mathbb{R} \) was arbitrary and so \( W_0(f) \) is covered completely by \( \epsilon \)-balls centered at \( T_{\delta_j} f \). Thus \( W_0(f) \) is precompact.

(\( \Leftarrow \)) Fix \( \epsilon > 0 \) and let \( O_1, ..., O_M \) be \( \epsilon/2 \)-balls that cover \( W_0(f) \). Without loss of generality, we can assume that \( O_j \cap W_0(f) \neq \emptyset \) for every \( j = 1, ..., M \). Therefore for every \( j = 1, ..., M \) we can pick some \( T_{y_j} f \in O_j \). Then, we are guaranteed that \( W_0(f) \subseteq \bigcup_{j=1}^M B(T_{y_j}, \epsilon) \).

We claim that every interval, \( J \), of length \( \Lambda = 2 \max_{1 \leq j \leq M} |y_j| \) contains an \( \epsilon \)-almost-period of \( f \). To see this, let \( y \) be the midpoint of the interval \( J \). Then since \( W_0(f) \subseteq \bigcup_{j=1}^M B(T_{y_j}, \epsilon) \), there must exist some \( y_0 \in \{1, ..., M\} \) such that \( \| T_y f - T_{y_0} f \|_{L^\infty(\mathbb{R})} < \epsilon \). Let \( \tau = y - y_0 \). Then by our choice of \( \Lambda \), we are guaranteed that \( \tau \in J \) and that \( \| T_{y+\tau} f - f \|_{L^\infty(\mathbb{R})} = \| T_{y_0+\tau} f - T_{y_0} f \|_{L^\infty(\mathbb{R})} < \epsilon \). This proves the claim.

All that is left to show is that \( f \) is almost-periodic is to show that \( f \) is continuous. We will show in fact that \( f \) is uniformly continuous, that is, \( \lim_{\delta \to 0} \| T_{\delta} f - f \|_{L^\infty(\mathbb{R})} = 0 \). We show this by contradiction; suppose that there exists some \( \epsilon > 0 \) and sequence of positive \( \delta_n \to 0 \) such that \( \| T_{\delta_n} f - f \|_{L^\infty(\mathbb{R})} \geq \epsilon \) for all \( n \). By precompactness of \( W_0(f) \), there exists a subsequence \( \{\delta_{n_k}\} \) such that \( T_{\delta_{n_k}} f \) converges in the \( L^\infty(\mathbb{R}) \) norm to some function \( g \) and we would necessarily have \( \| g - f \|_{L^\infty(\mathbb{R})} \geq \epsilon \). But it is an exercise in real analysis to show that \( f(x - \delta_{n_k}) \to f(x) \) in measure as \( \delta_{n_k} \to 0 \). Therefore, this gives \( f = g \) a.e. but this contradicts \( \| g - f \|_{L^\infty(\mathbb{R})} \geq \epsilon \). Therefore, \( f \) must be uniformly continuous; hence we have now shown that \( f \) is an almost-periodic function. \[\Box\]

Definition 3.8. The translation convex hull, \( W(f) \), of a function \( f \in L^\infty(\mathbb{R}) \) is the closed convex hull of \( \cup_{|a| \leq 1} W_0(af) \). Equivalently, it is the set of uniform limits of functions of the form

\[
\sum a_k T_{x_k} f, \quad x_k \in \mathbb{R}, \sum |a_k| \leq 1.
\]

Immediately from the definition, we have that \( W(f) \) is both convex and closed in \( L^\infty(\mathbb{R}) \).
Lemma 3.9. If $f$ is uniformly continuous we can define $W(f)$ as the closure of the set of all functions of the form $\varphi * f$ for $\varphi \in L^1(\mathbb{R})$ and $\|\varphi\|_{L^1(\mathbb{R})} \leq 1$

Proof. We will show that any finite sum $\sum_{k=1}^{N} a_k T_{x_k} f$ where $x_k \in \mathbb{R}$, $\sum_{k=1}^{N} |a_k| \leq 1$ can be approximated by a function of the form $\varphi * f$. Fix an $\epsilon > 0$. Let $\delta^i$ respond to the $\frac{\epsilon}{2N}$ challenge of continuity. Let $\gamma > 0$ be such that $|x_i - x_j| > \gamma$ for all $i \neq j$. Let $\delta = \min\{\delta^i, \gamma\}$. Therefore, the intervals $\{[x_k - \delta, x_k + \delta]\}_{k=1}^{N}$ are all pairwise disjoint. Let $\varphi(x) = \sum_{k=1}^{N} \frac{a_N}{2\delta} \mathbb{1}_{[x_k - \delta, x_k + \delta]}(x)$. We have that $\varphi \in L^1(\mathbb{R})$ since it is a finite linear combination of indicators of compact intervals. Then

$$\varphi \ast f(x) = \sum_{k=1}^{N} \int_{\mathbb{R}} \frac{a_N}{2\delta} \mathbb{1}_{[x_k - \delta, x_k + \delta]}(y) f(x - y) \, dy = \sum_{k=1}^{N} \frac{a_k}{2\delta} \int_{x_k - \delta}^{x_k + \delta} f(x - y) \, dy.$$ 

By the uniform continuity, we know that for all $y \in [x_k - \delta, x_k + \delta]$, $|f(x - y) - f(x - x_k)| < \frac{\epsilon}{2N}$. Therefore,

$$a_k(f(x - x_k) - \frac{\epsilon}{2N}) \leq \int_{x_k - \delta}^{x_k + \delta} \frac{a_k}{2\delta} f(x - y) \, dy \leq a_k(f(x - x_k) + \frac{\epsilon}{2N})$$

and thus for every $x \in \mathbb{R}$, $|\varphi \ast f(x) - \sum_{k=1}^{N} a_k T_{x_k} f(x)| \leq \sum_{k=1}^{N} \frac{a_k \epsilon}{2N} \leq \epsilon$.

Thus we have shown that functions of the form $\sum_{k=1}^{N} a_k T_{x_k} f$ where $x_k \in \mathbb{R}$, $\sum_{k=1}^{N} |a_k| \leq 1$, and the points $\{x_k\}$ do not accumulate can be approximated arbitrarily close in $L^\infty(\mathbb{R})$ by a function of the form $\varphi \ast f$. Since $\sum_{k=1}^{\infty} |a_k| \leq 1$, then for a general function of the form $\sum_{k=1}^{\infty} a_k T_{x_k} f$, repeat this proof for a large enough partial sum and the neglected tail will have contribution less than $\epsilon$.

For the other inclusion, suppose that we have $\varphi \ast f$ for some $\varphi \in L^1(\mathbb{R})$ with $\|\varphi\|_{L^1(\mathbb{R})} \leq 1$. For now, suppose that $\varphi$ is a simple function, i.e. that

$$\varphi = \sum_{i=1}^{N} c_k \mathbb{1}_{I_k}$$

where $\{I_k\}_{k=1}^{N}$ are disjoint intervals. Then we can write

$$\varphi \ast f(x) = \sum_{k=1}^{N} \int_{I_k} c_k f(x - y) \, dy.$$ 

Since $f$ (and therefore $c_k f$) is uniformly continuous, then the integral over $I_k$ can be approximated by the left Riemann integral. That is, fix some $\epsilon > 0$. Then for each $k = 1, ..., N$, we can partition $I_k$ into a finite number of points $\{x_{k,0}, x_{k,1}, ..., x_{k,M_k}\}$ such that

$$\left| \int_{I_k} c_k f(x - y) \, dy - \sum_{i=1}^{M_k} c_k f(x - x_{k,(i-1)}) \cdot (x_{k,i} - x_{k,(i-1)}) \right| < \epsilon/N.$$
for all $k = 1, \ldots, N$.

Then

$$\left| \varphi \ast f(x) - \sum_{k=1}^{N} \sum_{i=1}^{M_k} c_k f(x - x_{k,(i-1)}) \cdot (x_{k,i} - x_{k,(i-1)}) \right| < \epsilon.$$ 

By a simple relabeling of $\{x_{k,i}\}$ as $\{x_j\}$ and $\{c_k(x_{k,i} - x_{k,(i-1)})\}$ as $\{a_j\}$ we have

$$\left| \varphi \ast f(x) - \sum_{j} a_k f(x - x_j) \right| < \epsilon$$

as desired. We necessarily have $\sum_j |a_j| \leq 1$ since

$$1 \geq \|\varphi\|_{L^1(\mathbb{R})} = \sum_{k=1}^{N} |c_k|m(I_k) = \sum_{k=1}^{N} \sum_{i=1}^{M_k} |c_k|(x_{k,i} - x_{k,(i-1)}) = \sum_j |a_j|.$$ 

Now suppose $\varphi$ is any arbitrary function in $L^1(\mathbb{R})$ with $\|\varphi\|_{L^1(\mathbb{R})} \leq 1$. Then we can approximate $\varphi$ by a simple function $\varphi_N$ such that $\|\varphi - \varphi_N\|_{L^1(\mathbb{R})} < \frac{\epsilon}{2\|f\|_{L^\infty(\mathbb{R})}}$ (since the simple functions are dense in $L^1(\mathbb{R})$). Then follow the above proof to approximate $\varphi_N \ast f$ by $\sum_j a_j T_{x_j} f$ within $\epsilon/2$. Then

$$|\varphi \ast f - \sum_j a_j T_{x_j} f| \leq |\varphi \ast f - \varphi_N \ast f| + |\varphi_N \ast f - \sum_j a_j T_{x_j} f|$$

$$\leq \|f\|_{L^\infty(\mathbb{R})} \|\varphi - \varphi_N\|_{L^1(\mathbb{R})} + |\varphi_N \ast f - \sum_j a_j T_{x_j} f|$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon.$$ 

Lemma 3.10. For $f \in L^\infty(\mathbb{R})$

$$W(e^{2\pi i x f}) = \{e^{2\pi i x g} : g \in W(f)\}.$$ 

Proof. Let $g = \sum_k a_k T_{x_k} f \in W(f)$. Consider $e^{2\pi i x g}$. We can write

$$e^{2\pi i x g} = \sum_k a_k e^{2\pi i x T_{x_k} f} = \sum_k a_k e^{2\pi i (x - x_k)} e^{2\pi i x_k T_{x_k} f} = \sum_k a_k e^{2\pi i x_k T_{x_k} (e^{2\pi i x f}).}$$

By relabeling $b_k = a_k e^{2\pi i x_k}$, we’ve shown that $W(e^{2\pi i x f}) \supseteq \{e^{2\pi i x g} : g \in W(f)\}$. The proof for the opposite inclusion is almost identical in procedure. 

Lemma 3.11. $W(f)$ is compact iff $W_0(f)$ is precompact (iff $f \in AP(\mathbb{R})$).

Proof. ($\Rightarrow$): If $W(f)$ is compact, then there exists a finite cover of ball of radius $\epsilon > 0$. Since $W_0(f) \subseteq W(f)$, then that same finite collection of balls covers $W_0(f)$. This works for any $\epsilon > 0$, hence $W_0(f)$ is totally bounded.

($\Leftarrow$): If $W_0(f)$ is precompact, then for every $\epsilon > 0$ there exists a finite number of translates, $(T_{y_j} f)_{j=1}^{M}$ such that for any $y \in \mathbb{R}$, $\min_{1 \leq j \leq M} \|T_{y_j} f - T_{y_j} f\|_{L^\infty(\mathbb{R})} < \epsilon$.
Thus making $\epsilon/3$ every interval of length $\Lambda(\epsilon)$, which clearly gives $W(f)$ precompact. Since, by definition, $W(f)$ is closed, we have that it is compact. 

**Theorem 3.12.** $AP(\mathbb{R})$ is a closed subalgebra of $L^\infty(\mathbb{R})$.

**Proof.** We first show that if $f, g \in AP(\mathbb{R})$ then so is $f + g$. It is trivial to show that $W(f + g) \subseteq W(f) + W(g)$. Both $W(f)$ and $W(g)$ are compact and so $W(f + g)$ is precompact. But by definition, $W(f + g)$ is closed. Then by the previous lemma, $f + g \in AP(\mathbb{R})$.

Now to show that the product $fg \in AP(\mathbb{R})$. By Corollary 3.4, we have that $(f^2, g^2, (f + g)^2) \in AP(\mathbb{R})$. Therefore $fg = 1/2 ((f + g)^2 - f^2 - g^2) \in AP(\mathbb{R})$ as well. This shows that $AP(\mathbb{R})$ is a subalgebra of $L^\infty(\mathbb{R})$.

We now show that $AP(\mathbb{R})$ is closed. Consider a function $f$ in the closure of $AP(\mathbb{R})$. Since $f$ is the uniform limit of continuous functions (of form (3.1)), it is continuous. Since $f$ is in the closure of $AP(\mathbb{R})$, for any $\epsilon > 0$ there exists a $g \in AP(\mathbb{R})$ such that $\|f - g\|_{L^\infty(\mathbb{R})} < \epsilon/3$. If $\tau$ is an $\epsilon/3$-almost-period of $g$, then

$$T_\tau f - f = (T_\tau f - T_\tau g) + (T_\tau g - g) + (g - f)$$

which clearly gives $\|f_\tau - f\|_{L^\infty(\mathbb{R})} < \epsilon$. Hence, $\tau$ is an $\epsilon$-almost-period of $f$. Therefore every interval of length $\Lambda(\epsilon, g)$ contains an $\epsilon$-almost-period of $f$, thus making $f$ almost-periodic.

**Definition 3.13.** A trigonometric polynomial on $\mathbb{R}$ is a function of the form

$$f(x) = \sum_{j=1}^{n} a_j e^{2\pi i \xi_j x}, \quad \xi_j \in \mathbb{R}.$$ 

We call the numbers $\xi_j$ the frequencies of $f$.

One can see that for any $j$, $a_j e^{2\pi i \xi_j x}$ is periodic, hence almost-periodic. Then by the previous theorem, the above sum is also almost-periodic. Moreover, all uniform limits of trigonometric polynomials are almost-periodic. We will lead up to a theorem that states that every almost-periodic function is the uniform limit of trigonometric polynomials.
Definition 3.14. The norm spectrum of a function $h \in L^\infty(\mathbb{R})$ is the set

$$
\sigma(h) = \{ \xi \in \mathbb{R} : ae^{2\pi i \xi x} \in W(h) \text{ for sufficiently small } a \neq 0 \}.
$$

It may well be that $\sigma(h)$ is empty, even if $h \neq 0$. For example, if $h \in C_0(\mathbb{R})$ then every function in the translation convex hull, that is functions of the form (3.1), is also in $C_0(\mathbb{R})$. So $W(h) \subset C_0(\mathbb{R})$ and therefore $\sigma(h) = \emptyset$.

Remark 3.15. Notice that by (3.2) we have $\sigma(e^{2\pi i \xi x} h) = \{ \gamma \in \mathbb{R} : ae^{2\pi i \gamma x} \in W(e^{2\pi i \xi x} h) \} = \{ \gamma \in \mathbb{R} : ae^{2\pi i (\gamma - \xi) x} \in W(h) \}$.

Lemma 3.16. If $h \in L^\infty(\mathbb{R})$, then $\sigma(h) \subseteq \text{supp}(h)$.

Proof. Notice that for $y \in \mathbb{R}$, $\widehat{T_yh}(\xi) = e^{2\pi i \xi y} \widehat{h}(\xi)$. Therefore, $\text{supp}(\widehat{T_yh}) = \text{supp}(\widehat{h})$. Therefore, for any $f \in W(h)$ (and hence of the form (3.1)) then $\text{supp}(\hat{f}) \subseteq \text{supp}(\hat{h})$.

Now suppose $f \in W(h)$ and that $f = ae^{2\pi i \xi x}$, then a simple calculation reveals that $\hat{f} = a\delta_{\xi}$ and therefore $\text{supp}(\hat{f}) = \{ \xi \}$. This proves that if $\xi \in \sigma(h)$ then $\xi \in \text{supp}(h)$. \qed

Lemma 3.17. Assume that $h \in L^\infty(\mathbb{R})$ is uniformly continuous and that $g_n = \eta K(\eta x) * h$ converges uniformly to a nonzero limit as $\eta \to 0$ (where $K$ is the usual Féjer kernel). Then $0 \in \sigma(h)$.

Proof. Since $g_n = \eta K(\eta x) * h$ then $\hat{g}_n = \eta \hat{K}(\eta x) \hat{h}$. A simple calculation reveals that

$$
\eta \hat{K}(\eta x)(\xi) = \int_{\mathbb{R}} \eta K(\eta x) e^{-2\pi i \xi x} \, dx = \int_{\mathbb{R}} K(x) e^{-2\pi i \xi / \eta x} \, dx = \hat{K} \left( \frac{\xi}{\eta} \right).
$$

Then by properties of the Féjer kernel, we have that $\text{supp}(\hat{g}_n) \subseteq [-\eta, \eta]$ and so letting $g = \lim_{\eta \to 0} g_n$ we have $\text{supp}(\hat{g}) = \{ 0 \}$. By Theorem 2.11 and Exercise 2.15 we have that $g$ is a (nonzero) constant function.

By Lemma 3.9 both $g_n$ and $g$ are in $W(f)$ for all $\eta$ (This is where we use the uniform continuity of $h$). Since $g$ is a constant function, that implies that $0 \in \sigma(h)$. \qed

Corollary 3.18. Let $\mu$ be a bounded measure on $\mathbb{R}$ and assume $\mu(\{ 0 \}) \neq 0$. Let $h(x) = \int e^{2\pi i \xi x} \, d\mu(x)$ (then $\mu = \hat{h}$); then $0 \in \sigma(h)$.

Proof. By following the proof and notation in the previous lemma, we have $\hat{g}_n = \hat{K}(\xi / \eta) \hat{\mu}$ and therefore $\hat{g}_n$ tends to $\mu(\{ 0 \}) \delta_0$ in $M_t(\mathbb{R})$ which implies that $g_n \to \mu(\{ 0 \})$ uniformly. \qed

Note that there is no specific role of $0 \in \mathbb{R}$ in the above result. That is, if $\mu(\{ \xi \}) \neq 0$, then $\xi \in \sigma(h)$ where $h$ is as above. Also, it was not essential to use the Féjer kernel in the proof. In fact, if $F \in L^1(\mathbb{R})$ and $\eta F(\eta x) * h$ converges uniformly to a nonvanishing limit, then $0 \in \sigma(h)$. We omit the detail of the proof here but they can be found in [Katz].
Lemma 3.19. Let \( f \in AP(\mathbb{R}) \) and assume \( 0 \notin \sigma(f) \). Then for every \( F \in L^1(\mathbb{R}) \), \( \lim_{n \to 0} \| \eta F(\eta x) \ast f \|_{L^\infty(\mathbb{R})} = 0 \).

**Proof.** Without loss of generality, assume that \( \| F \|_{L^1(\mathbb{R})} \leq 1 \). By Lemma 3.9, \( \eta F(\eta x) \ast f \in W(f) \). If \( \eta F(\eta x) \ast f \) did not tend to 0 as \( \eta \to 0 \), then, since \( W(f) \) is compact, it would have a nonzero limit point. Then by the above remark (proof omitted), \( 0 \in \sigma(f) \).

Lemma 3.19 has the following converse:

**Lemma 3.20.** Let \( f \in AP(\mathbb{R}) \) and \( F \in L^1(\mathbb{R}) \) with \( \int \mathbb{R} F(x) \, dx \neq 0 \). If for some sequence \( \eta_n \to 0 \), \( \lim_{n \to \infty} \| \eta_n F(\eta_n x) \ast g \|_{L^\infty(\mathbb{R})} = 0 \), then \( 0 \notin \sigma(f) \).

**Proof.** First notice that if \( g \) is a translate of \( f \), then by the translation invariance of the convolution, \( \lim_{n \to \infty} \| \eta_n F(\eta_n x) \ast g \|_{L^\infty(\mathbb{R})} = 0 \) as well. Thus any linear combination, and hence all elements \( g \in W(f) \) satisfy that estimate as well. If \( g = c \) for some constant, \( c \), then \( \eta_n F(\eta_n x) \ast g = c \int \mathbb{R} \eta_n F(\eta_n x) \, dx = c F(0) \). Since we assumed \( F(0) \neq 0 \), this implies that \( c = 0 \) hence the only constant in \( W(f) \) is zero. Therefore \( 0 \notin \sigma(f) \).

**Theorem 3.21.** For every \( f \in AP(\mathbb{R}) \) there exists a unique number, \( M(f) \), called the mean value of \( f \), having the property that \( 0 \in \sigma(f - M(f)) \).

**Proof.** In the proof of Lemma 3.17 that \( \eta K(\eta x) \ast f \) converges uniformly as \( \eta \to 0 \) to a constant function, The function \( \eta K(\eta x) \ast f \in W(f) \) by Lemma 3.9. Then since \( W(f) \) is compact, there exists a number \( \alpha \) such that for an appropriate sequence \( \{ \eta_n \} \) with \( \lim_{n \to \infty} \eta_n = 0 \), we have \( \eta_n K(\eta_n x) \ast f \) converges uniformly to \( \alpha \) as \( n \to \infty \).

Since \( K(0) = 1 \) and \( \eta_n K(\eta_n x) \ast (f - \alpha) \to 0 \) uniformly, then by 3.20 we have \( 0 \notin \sigma(f - \alpha) \).

Now to show the uniqueness of \( \alpha \). If \( \beta \) is another number with \( 0 \notin \sigma(f - \beta) \), then by Lemma 3.19

\[
\eta K(\eta x) \ast [(f - \alpha) - (f - \beta)] = \eta K(\eta x) \ast (f - \alpha) - \eta K(\eta x) \ast (f - \beta)
\]

converges uniformly to zero as \( \eta \to 0 \). But \( [(f - \alpha) - (f - \beta)] = \beta - \alpha \), hence \( \eta K(\eta x) \ast (\beta - \alpha) \to 0 \) which implies \( \beta = \alpha \).

By following the proofs to Theorem 3.21 and Lemma 3.20 we obtain the following corollary:

**Corollary 3.22.** If \( f \in AP(\mathbb{R}) \) and \( F \in L^1(\mathbb{R}) \) \( \{ F \neq 0 \} \) then \( \eta F(\eta x) \ast f \) converges uniformly as \( \eta \to 0 \) to the constant \( \tilde{F}(0) M(f) \).

**Corollary 3.23.** For \( f \in AP(\mathbb{R}) \), we have

\[
M(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x) \, dx.
\]

**Proof.** Let \( F(x) = 1/2 \) for \(-1 < x < 1\) and 0 otherwise and let \( \eta = \frac{1}{T} \). Then we can compute \( \tilde{F}(0) = 1 \) and the convolution at the origin:

\[
\eta F(\eta x) \ast f(0) = \int_{\mathbb{R}} \frac{1}{T} F(0 - y) f(y) \, dy = \frac{1}{2T} \int_{-T}^{T} f(y) \, dy.
\]
Then since $\eta F(\eta x) * f$ converges uniformly as $\eta \to 0$ to the constant $\tilde{F}(0)M(f)$ we obtain the desired result. \hfill \Box

By Remark 3.15 we can conclude that $\xi \in \sigma(f)$ iff $0 \in \sigma(f e^{-2\pi \xi x})$. Consequently

$$\xi \in \sigma(f) \text{ iff } M(f e^{-2\pi \xi x}) \neq 0.$$  

By Corollary 3.18, if $\hat{f}$ is a measure then $\hat{f}(\{0\}) = M(f)$ and similarly $\hat{f}(\{\xi\}) = M(f e^{-2\pi \xi x})$. Therefore, by using the mean value, we are able to recover all discrete parts of $\hat{f}$. It turns out that if $f \in \text{AP}(\mathbb{R})$, then $\hat{f}$ has no continuous part.

We can easily verify that the mean value has all the nice properties of a translation invariant integral. That is, for $f, g \in \text{AP}(\mathbb{R})$,

$$M(f + g) = M(f) + M(g),$$  

$$M(af) = aM(f),$$  

$$M(T_y f) = M(f).$$

Moreover, we have that the mean value is positive:

**Lemma 3.24.** Assume $f \in \text{AP}(\mathbb{R})$, $f(x) \geq 0$, and $f$ not identically zero. Then $M(f) > 0$.

**Proof.** By the translation invariance of the mean value, we may assume, without loss of generality that $f(0) > 0$. In addition, by Lemma 3.5, since $f$ is uniformly continuous, there exists some $\alpha > 0$ such that $f(x) > \alpha$ for every $|x| < \alpha$. Let $\Lambda = \Lambda(\alpha/2, f)$. Then every interval in $\mathbb{R}$ of length $\Lambda$ contains an $\alpha/2$-almost-period of $f$, call it $\tau$. Then $f(x) > \alpha/2$ on $(\tau - \alpha, \tau + \alpha)$. This implies that the integral of $f$ over any interval of length $\Lambda$ is at least $\alpha^2$. Therefore $M(f)$ must be at least $\alpha^2/\Lambda$. \hfill \Box

On the space $\text{AP}(\mathbb{R})$ we can define an inner product by

$$\langle f, g \rangle_M = M(\bar{f}g).$$  

The bilinearity of the above inner product is obvious. For $f \neq 0$ we have $\langle f, f \rangle_M > 0$ by the previous lemma. In fact, with this inner product, the space $\text{AP}(\mathbb{R})$ satisfies all properties of a Hilbert space except completeness. We say that $\text{AP}(\mathbb{R})$ equipped with this inner product is a pre-Hilbert space.

The exponentials $\{e^{2\pi i \xi x}\}_{\xi \in \mathbb{R}}$ form an orthonormal family since

$$\langle e^{2\pi i \xi x}, e^{2\pi i \eta x} \rangle_M = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{2\pi i (\xi - \eta)x} \, dx = \begin{cases} 1 & \text{if } \xi = \eta, \\ 0 & \text{if } \xi \neq \eta. \end{cases}$$

As a matter of notation we set

$$\hat{f}(\{\xi\}) = \langle f, e^{2\pi i \xi x} \rangle_M = M(f e^{-2\pi \xi x}).$$

Then Bessel’s inequality gives

$$\sum_{\xi \in \mathbb{R}} |\hat{f}(\{\xi\})|^2 \leq \langle f, f \rangle_M = M(|f|^2)$$
and therefore \( \hat{f}(\{\xi\}) = 0 \) except for at most countably many \( \xi \in \hat{\mathbb{R}} \). This justifies the assertion that \( \hat{f} \) has no continuous part after the proof of Corollary 3.23.

We know that the exponentials \( \{e^{2\pi i \xi x}\} \) are an orthonormal family but we have not made any assertion about completeness. Completeness of an orthonormal family is equivalent to the uniqueness theorem; that is, an orthonormal family is complete if zero is the only element all of whose Fourier coefficients vanish. We state and prove the result below:

**Theorem 3.25** (Uniqueness Theorem). Let \( f \in AP(\mathbb{R}), f \neq 0 \). Then \( \sigma(f) \neq \emptyset \).

Before we prove Theorem 3.25, recall the theorems of Wiener and Bochner:

**Theorem 3.26** (Wiener’s Theorem). Let \( \mu \in M_b(\mathbb{R}) \). Then

\[
\sum |\mu(\{x\})|^2 = \lim_{\lambda \to \infty} \frac{1}{2\lambda} \int_{-\lambda}^{\lambda} |\hat{\mu}(\xi)|^2 d\xi.
\]

In particular, \( \mu \) is continuous iff the integral on the right-hand-side equals zero.

**Theorem 3.27** (Bochner’s Theorem). A function \( \varphi \) on \( \hat{\mathbb{R}} \) is a Fourier-Stieltjes transform of a positive measure iff it is positive definite and continuous.

**Proof of Theorem 3.25.** First suppose that \( \hat{f} \) is a measure on \( \hat{\mathbb{R}} \). Suppose \( \sigma(f) \neq \emptyset \); that is, suppose \( \xi \in \sigma(f) \) iff \( M(f e^{-2\pi i \xi x}) \neq 0 \). Then by the remarks after Corollary 3.23, \( \hat{f} \) has discrete part at \( \xi \). Thus, saying \( \sigma(f) \neq \emptyset \) is equivalent to saying that \( \hat{f} \) has a nonvanishing discrete part. By Wiener’s theorem, if \( \hat{f} \) is continuous, then \( \lim_{T \to \infty} (2T)^{-1} \int_{-T}^{T} |f(x)|^2 dx = 0 \). Then by Lemma 3.24, \( f \equiv 0 \), a contradiction. Therefore, \( M(f) \neq 0 \).

Suppose now that \( f \in AP(\mathbb{R}) \). We claim that there always exists a nonzero function \( h \in W(f) \) such that \( \hat{h} \) is a positive measure on \( \hat{\mathbb{R}} \). Then we would by done by the previous argument and the fact that for any \( h \in W(f) \), then \( \sigma(h) \subseteq \sigma(f) \).

Construct \( h \) as follows: without loss of generality, \( \|f\|_{L^\infty(\mathbb{R})} \leq 1 \). Set

\[
h_T(x) = \frac{1}{2T} \int_{-T}^{T} f(y) f(x+y) \, dy.
\]

By Lemma 3.9 \( h_T \in W(f) \) for every \( T > 0 \). Since \( W(f) \) is compact, let \( h \) denote a limit point of \( h_T \); that is, for some appropriate subsequence \( \{T_n\} \), we have \( h = \lim_{n \to \infty} h_{T_n} \). Then \( h(0) = \lim_{n \to \infty} h_{T_n}(0) = M(|f|^2) \neq 0 \), therefore \( h \neq 0 \).

By our remarks, \( h \) is continuous; we wish to show now that it is also positive definite: Let \( x_j \in \mathbb{R} \) and \( z_j \in \mathbb{C} \) for \( j = 1, \ldots, N \), then

\[
\sum_{j,k=1}^{N} h(x_j - x_k) z_j \bar{z}_k = \lim_{n \to \infty} \frac{1}{2T_n} \int_{-T_n}^{T_n} \sum_{j,k=1}^{N} f(x_j + y) f(x_k + y) z_j \bar{z}_k \, dy
\]

\[
= \lim_{n \to \infty} \frac{1}{2T_n} \int_{-T_n}^{T_n} \sum_{j=1}^{N} z_j f(x_j + y) \, dy \geq 0.
\]

So then by Bochner’s theorem, \( \hat{h} \) is a positive measure, and we’re done. \( \square \)
Lemma 3.9. \( f_1 \) is the uniform limit of \( \{ \hat{f} \} \).

Proof. Let us assume, without loss of generality, that \( \mu \equiv 0 \). Then by Lemma 3.9, \( f \) is almost-periodic. Also, if \( M(|g|) \leq 1 \), then \( f * g \in W(f) \). Furthermore, if \( g \) is a trigonometric polynomial then \( (f * g)(x) = \sum \hat{f}(|\xi|) \hat{g}(\xi) e^{2\pi i \xi x} \).

Proof. Let us assume, without loss of generality, that \( \mu \equiv 0 \). Then by Lemma 3.9, \( f \) is almost-periodic. Also, if \( M(|g|) \leq 1 \), then \( f * g \in W(f) \). Furthermore, if \( g \) is a trigonometric polynomial then \( (f * g)(x) = \sum \hat{f}(|\xi|) \hat{g}(\xi) e^{2\pi i \xi x} \).

Corollary 3.28. For \( f \in AP(\mathbb{R}) \), \( \sum |\hat{f}(|\xi|)|^2 = M(|f|^2) \).

Theorem 3.29. If \( f \in AP(\mathbb{R}) \) and \( \hat{f} \in M_b(\mathbb{R}) \), then

i. \( \hat{f} = \sum \hat{f}(|\xi|) \delta_\xi \)

ii. \( \| f \|_{M_1(\mathbb{R})} = \sum |\hat{f}(|\xi|)| \)

iii. \( f(x) = \sum \hat{f}(|\xi|) e^{2\pi i \xi x} \).

Proof. Since \( \hat{f}(|\xi|) = M(f e^{-2\pi i \xi x}) \), then \( \sum \hat{f}(|\xi|) \delta_\xi \) is just the discrete part of \( \hat{f} \). Therefore, we have one inequality of iii., namely \( \| f \|_{M_1(\mathbb{R})} \leq \sum |\hat{f}(|\xi|)| \).

Let \( \mu \) denote the continuous part of \( \hat{f} \), then \( \mu \) is the Fourier transform of the function \( f - \sum \hat{f}(|\xi|) e^{2\pi i \xi x} \in AP(\mathbb{R}) \) since \( AP(\mathbb{R}) \) is a closed subalgebra of \( L^\infty(\mathbb{R}) \) (Theorem 3.12). By following the proof of the Uniqueness theorem, we see that \( \mu \equiv 0 \). Therefore, \( f - \sum \hat{f}(|\xi|) e^{2\pi i \xi x} \) and \( \| f \|_{M_1(\mathbb{R})} = \sum |\hat{f}(|\xi|)| \).

Thus far, we’ve shown that for \( f \in AP(\mathbb{R}) \), the Fourier series of \( f \), \( \sum \hat{f}(|\xi|) e^{2\pi i \xi x} \), converges to \( f \) in the norm induced by the bilinear form \( \langle \cdot, \cdot \rangle_M \). We now look to show that the convergence also happens in the \( L^\infty(\mathbb{R}) \) norm, i.e., uniformly. This fact is already known for Fourier series of periodic functions. Before we show this result, we require some definitions and results:

Definition 3.30. For \( f, g \in AP(\mathbb{R}) \), we define the mean convolution of the two almost-periodic functions as

\[
(f * M g)(x) = M_g (f(x - y)g(y)) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^T f(x - y)g(y) \, dy.
\]

Notice that the function \( h \) introduced in the proof of the uniqueness theorem (Theorem 3.25) is nothing more than \( f * M f(-x) \).

Lemma 3.31. If \( f, g \in AP(\mathbb{R}) \) then \( f * M g \) is almost-periodic. Also, if \( M(|g|) \leq 1 \), then \( f * M g \in W(f) \). Furthermore, if \( g \) is a trigonometric polynomial then \( (f * M g)(x) = \sum \hat{f}(|\xi|) \hat{g}(|\xi|) e^{2\pi i \xi x} \).

Proof. Let us assume, without loss of generality, that \( M(|g|) < 1 \). Then by Lemma 3.9, \( f * g \in W(f) \). Since \( W(f) \) is compact and the definition of \( f * M g \) is the uniform limit of \( \frac{1}{2T} \int_{-T}^T f(x - y)g(y) \, dy \), then \( f * M g \in AP(\mathbb{R}) \).

If \( g(x) = \sum \hat{g}(|\xi|) e^{2\pi i \xi x} < \infty \) for all \( x \), then \( f * M g = \sum \hat{g}(|\xi|) f * M e^{2\pi i \xi x} \) and since

\[
M_g(f(x - y)e^{2\pi i \xi y}) = M(f(t)e^{2\pi i \xi (x-t)}) = \hat{f}(|\xi|) e^{2\pi i \xi x}
\]

we’re done.
Lemma 3.32. Given a finite number of points \( \xi_1, \ldots, \xi_N \in \mathbb{R} \) and \( \epsilon > 0 \), then there exists a trigonometric polynomial \( B \) with the following properties:

a. \( B(x) \geq 0 \)

b. \( M(B) = 1 \)

c. \( \hat{B}(\{\xi_j\}) > 1 - \epsilon \) for \( j = 1, \ldots, N \).

Proof. First notice that if \( \xi_1, \ldots, \xi_N \) are all integers, then for any integer \( m > \frac{1}{\epsilon} \max_{j=1,\ldots,N} |\xi_j| \), the Féjer kernel \( K_m = \sum_{k=-m}^{m} \left( 1 - \frac{|k|}{m+1} \right) e^{2\pi ikx} \) has all of the desired properties.

For the general case, let \( \lambda_1, \ldots, \lambda_q \) be a basis for \( \{\xi_j\}_{j=1}^N \); that is, \( \lambda_1, \ldots, \lambda_q \) are linearly independent over the rational numbers and each \( \xi_j \) can be written \( \xi_j = \sum_{k=1}^{q} A_{j,k} \lambda_k \). Let \( \epsilon_1 > 0 \) be such that \( (1 - \epsilon_1)^q > 1 - \epsilon \), and let \( m > (\epsilon_1)^{-1} \max_{j,k} |A_{j,k}| \). We claim that \( B = \prod_{k=1}^{q} K_m(\lambda_k x) \) satisfies all the required properties.

Firstly, Property a is apparent since \( B \) is the product of nonnegative functions. In order to verify b and c, we rewrite \( B \) as

\[
B(x) = \sum_{k=1}^{q} |A_{1,k}| \cdot \cdots \cdot |A_{q,k}| \leq m \left( 1 - \frac{|k_1|}{m+1} \right) \cdots \left( 1 - \frac{|k_q|}{m+1} \right) e^{2\pi i(k_1 \lambda_1 + \cdots + k_q \lambda_q)x}
\]

Since we chose the collection \( \{\lambda_j\}_{j=1}^q \) to be linearly independent, there is no rearranging of terms to have to same frequency in (3.5). Therefore, \( \hat{B}(\{0\}) \) must equal the constant term in (3.5), namely 1. This establishes property b. Finally,

\[
\hat{B}(\{\xi_j\}) = \hat{B}(\sum_{k=1}^{q} A_{j,k} \lambda_k) = \prod_{k=1}^{q} \left( 1 - \frac{|A_{j,k}|}{m+1} \right) > (1 - \epsilon_1)^q > 1 - \epsilon
\]

which establishes property c. \( \square \)

We are now able to prove the result we set out to show:

Theorem 3.33. Let \( f \in AP(\mathbb{R}) \). Then \( f \) can be approximated uniformly by trigonometric polynomials \( P_n \in W(f) \).

Proof. Since \( \sigma(f) \) is countable, denote it by \( \{\xi_j\}_{j=1}^\infty \). For each natural number \( n \), let \( B_n \) be the trigonometric polynomial described in the preceding lemma for \( \xi_1, \ldots, \xi_n \) and \( \epsilon = 1/n \). Write \( P_n = f * B_n \). By Lemma 3.31, \( P_n \in W(f) \) and by part c of Lemma 3.32, \( \lim_{n \to \infty} \hat{P}_n(\{\xi_j\}) = \hat{f}(\{\xi_j\}) \) for every \( \xi_j \in \sigma(f) \). If \( \xi \notin \sigma(f) \), then \( \hat{P}_n(\{\xi\}) = \hat{f}(\{\xi\}) = 0 \) for all \( n \). Therefore, if \( g \) is a limit point of \( P_n \in W(f) \), then \( \hat{g}(\{\xi\}) = \hat{f}(\{\xi\}) \) for all \( \xi \). Then by the uniqueness theorem, \( g = f \). Therefore, \( f \) is the only limit point of the sequence \( \{P_n\} \) in the compact space \( W(f) \). Therefore, \( P_n \) converges to \( f \) in norm. \( \square \)
4 The Paley-Wiener Theorems

The purpose of this section is to study the relationship between properties of analyticity and growth of a function on \(\mathbb{R}\), and the growth of its Fourier transform in \(\hat{\mathbb{R}}\). In particular, in which situations can \(\hat{f}\) be viewed as an analytic function in some region in the complex plane. It should be noted that much of structure of this section was obtained from [Rud].

Consider \(\mathbb{R}\) as the real axis in the complex plane \(\mathbb{C}\). It is clear that \(f\) is analytic on \(\mathbb{R}\) iff it is the restriction to \(\mathbb{R}\) of some \(F\) analytic in a domain containing \(\mathbb{R}\), that is \(\{z = x + iy : y = 0\}\). However, the issue here is that the domain need not be nice. We cannot say that \(F\) is analytic on a strip \(\{z = x + iy : |y| < a\}\) for some \(a > 0\). Moreover, \(F\) need not be bounded in strips around \(\mathbb{R}\), nor on \(\mathbb{R}\) itself. An example of such a function is illustrated in the following exercise from [Katz]:

**Exercise 4.1** (Katznelson IV.7.1). Show that \(F(z) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{(z+n)^2 + 1/n}\) is analytic on \(\mathbb{R}\) and \(F|_{\mathbb{R}} \in L^1 \cap L^\infty(\mathbb{R})\); however, \(F\) is not holomorphic in any strip \(\{z \in \mathbb{C} : |\Im(z)| < a\}\) for any \(a > 0\).

**Proof.** Some elementary estimates give us

\[
|F(x)| = F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{(x+n)^2 + 1/n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{1/n} = 2
\]

which proves that \(F(z)|_{\mathbb{R}} \in L^\infty(\mathbb{R})\). So \(F(z)|_{\mathbb{R}}\) is bounded; to show \(F(z)|_{\mathbb{R}} \in L^1(\mathbb{R})\) we just need to show that the tails of \(F(x)\) decay faster than some \(L^1(\mathbb{R})\) function. This is clear since

\[
|F(x)| = F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{(x+n)^2 + 1/n} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{1}{x^2} = \frac{1}{x^2}
\]

which is indeed integrable away from zero.

As a real function, \(F(x)\) is infinitely differentiable (analytic) since we can differentiate term by term. However, as a complex function, \(F(z)\) is not analytic.

We can equivalently write \(F(z)\) as:

\[
F(x + iy) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{x^2 - y^2 + 1/n - 2ixy}{1/n^2 + 2x^2/n + x^4 - 2y^2/n + 2x^2y^2 + y^4}.
\]

One can show that the imaginary part diverges at \(x = 0\) for any \(y \neq 0\) (see figure). 

We shall describe two classes of analytic functions that arise from the Fourier transform.

For the first class, take some \(F \in L^2(\mathbb{R})\) that vanishes on \((-\infty, 0]\). In other words, pick \(F \in L^2((0, \infty))\). Define on the upper-half plane \(\Pi^+ = \{z = x + iy : y > 0\}\),

\[
f(z) = \int_0^{\infty} F(t)e^{2\piitz} dt, \quad z \in \Pi^+
\]

(4.1)
Figure 1: The restriction of $F(z)$ to the real line (left) and a graph of $|F(z)|$ (right). We can see that restricted to $\mathbb{R}$, $F$ is smooth, bounded, and decays well enough. However in the complex plane, the function (in modulus) is unbounded since the imaginary part of $F$ diverges at $x = 0$ (below).

Notice that if $z \in \Pi^+$, then $|e^{2\pi itz}| = |e^{2\pi i(x+iy)}| = |e^{-2\pi ty}|$. Therefore, by Hölder

$$|f(z)| = \left| \int_0^\infty F(t)e^{2\pi itz} \, dt \right| \leq \|F\|_{L^2((0,\infty))} \|e^{-2\pi ty}\|_{L^2((0,\infty))}$$

which is finite since $y > 0$. So we can say that as defined by (4.1), $f(z)$ exists for all $z \in \Pi^+$.

Now suppose that $\{z_n\}_{n=1}^\infty$ is a sequence in $\Pi^+$ that converges to $z \in \Pi^+$. Furthermore, assume that for every $n \geq 1$, we have $\text{Im}(z_n), \text{Im}(z) > \delta > 0$. Then by Lebesgue’s dominated convergence, $\lim_{n \to \infty} \int_0^\infty |e^{2\pi itz_n} - e^{2\pi itz}|^2 \, dt = 0$ since the integrand is bounded above by $4e^{-4\pi \delta t}$ which is integrable. Then by Cauchy-Schwartz, we get

$$|f(z) - f(z_n)| = \left| \int_0^\infty F(t) \left( e^{2\pi itz_n} - e^{2\pi itz} \right) \, dt \right| \leq \|F\|_{L^2} \|e^{2\pi itz_n} - e^{2\pi itz}\|_{L^2} \to 0$$

that is, $f$ is continuous on $\Pi^+$. Notice that if $\gamma$ is a finite closed curve in $\Pi^+$, then

$$\left| \int_\gamma f(z) \, dz \right| = \left| \int_\gamma \int_0^\infty F(t)e^{2\pi itz} \, dt \, dz \right| \leq |\gamma| \max_{\gamma} |f|$$

which is necessarily finite since we’ve shown $f$ is continuous. If we interchange
the order of the integrals, then
\[ \int_0^\infty \int_\gamma F(t) e^{2\pi itz} dt \, dz = \int_0^\infty F(t) \int_\gamma e^{2\pi itz} dz \, dt = 0 \]
since \( e^{2\pi itz} \) is an analytic function of \( z \). Then by Fubini’s Theorem, since both quantities are finite, they are equal and \( \int_\gamma f(z) \, dz = 0 \). Then by Morera’s theorem, \( f \) is analytic in \( \Pi^+ \).

Now if we rewrite \( f \) as \( f(x + iy) = \int_0^\infty F(t) e^{-2\pi ty} e^{2\pi itx} dt \) and fix \( y > 0 \), then by Plancherel,
\[ \int_\mathbb{R} |f(x + iy)|^2 \, dx = \int_0^\infty |F(t)|^2 e^{-4\pi ty} dt \leq \int_0^\infty |F(t)|^2 \, dt. \]

Therefore, we have proven the following result:

(a) If \( f \) is of the form given in (4.1), then \( f \) is holomorphic in \( \Pi^+ \) and \( f \) restricted to horizontal lines in \( \Pi^+ \) forms a bounded family in \( L^2(\mathbb{R}) \).

The second class of functions that we consider are of the form
\[ (4.2) \quad f(z) = \int_{-A}^A F(t) e^{2\pi itz} dt \]
where \( 0 < A < \infty \) is fixed and \( F \in L^2(-A,A) \). Following the same proof as in the first class shows that \( f \) is analytic (this time in all of \( \mathbb{C} \)). Moreover, \(|f(z)| \leq \int_{-A}^A |F(t)| e^{-2\pi ty} dt \leq e^{2\pi A|z|} \int_{-A}^A |F(t)| \, dt \). Since \( L^2(\mathbb{R}) \subseteq L^1_{\text{loc}}(\mathbb{R}) \), then this last integral is finite, call it \( C \). Therefore \(|f(z)| \leq Ce^{A|z|} \). We call such functions of this form to be of exponential type.

Therefore, we have proven the following result:

(b) If \( f \) is of the form given in (4.2), then \( f \) is an entire function and of exponential type. Moreover the restriction of \( f \) to \( \mathbb{R} \) is in \( L^2(\mathbb{R}) \).

It turns out that the converses to both of our results, (a) and (b), also are true. This is the content of the Paley-Wiener theorems.

**Theorem 4.2** (Paley-Wiener). *Suppose \( f \) is analytic in \( \Pi^+ \) and suppose that*
\[ \sup_{0 < y < \infty} \int_\mathbb{R} |f(x + iy)|^2 \, dx = C < \infty. \]
*Then there exists an \( F \in L^2((0,\infty)) \) such that*
\[ (4.3) \quad f(z) = \int_0^\infty F(t) e^{2\pi itz} \, dt, \quad \forall z \in \Pi^+ \]
*and*
\[ (4.4) \quad \int_0^\infty |F(t)|^2 \, dt = C \]
Proof. Fix $0 < y < \infty$. For any $\alpha > 0$ let $\Gamma_\alpha$ be the rectangular path with vertices $\pm \alpha + i$ and $\pm \alpha + iy$. Since $f(z)e^{-2\pi i tz}$ is analytic in $\Pi^+$, then by Cauchy’s integral theorem, $\int_{\Gamma_\alpha} f(z)e^{-2\pi i tz} \, dz = 0$.

For $t, \beta$ real, let $\Phi(\beta)$ denote the integral of $f(z)e^{-2\pi i tz}$ along the line connecting $\beta + i$ and $\beta + iy$. Let $I = [y, 1]$ if $y < 1$ or $I = [1, y]$ if $y > 1$. Then

$$|\Phi(\beta)|^2 = \left| \int_I f(\beta + iu)e^{-2\pi i(\beta + iu)} \, du \right|^2 \leq \left( \int_I |f(\beta + iu)|^2 \, du \right) \left( \int_I |e^{-4\pi i(\beta + iu)}| \, du \right) = \left( \int_I |f(\beta + iu)|^2 \, du \right) \left( \int_I e^{4\pi u} \, du \right).$$

Denote the first of these integrals, by $\Lambda(\beta) = \int_I |f(\beta + iu)|^2 \, du$. Then by (4.3), $\int_{-\infty}^\infty \Lambda(\beta) \, d\beta \leq Cm(I)$ where the last term is the Lebesgue measure of the interval $I$.

We claim that since $\Lambda$ is continuous and integrable, then there exists an increasing sequence of real numbers $\{\alpha_j\}_{j=1}^\infty$ with $\alpha_j \to \infty$ such that $\lim_{j \to \infty} \Lambda(\alpha_j) = \lim_{j \to \infty} \Lambda(-\alpha_j) = 0$. To see this, fix any $\epsilon > 0$ and $A > 0$ and suppose there does not exist any $x > A$ such that neither $|\Lambda(x)| < \epsilon$ nor $|\Lambda(-x)| < \epsilon$. Then $\infty > \|\Lambda\|_1 \geq \int_{-\infty}^\infty A + \int_\infty^\infty |\Lambda(x)| \, dx \geq c m((A, \infty))$ which is infinite, a contradiction. Therefore, pick any $\alpha_0 > 0$. Then inductively, for any integer $n \geq 1$, we can select some $\alpha_n > \alpha_{n-1}(=A)$ such that $|\Lambda(\alpha_n)| < 1/n$ and $|\Lambda(-\alpha_n)| < 1/n$. Then since we’ve shown that $|\Phi(\beta)|^2 \leq \Lambda(\beta) \int_I e^{4\pi iu} \, du$, then we also have

$$\lim_{j \to \infty} \Phi(\alpha_j) = 0, \quad \lim_{j \to \infty} \Phi(-\alpha_j) = 0.$$

Notice that the proof so far holds for any real $t$ and the sequence $\{\alpha_j\}$ is valid for any such $t$.

Now define

$$g_j(y, t) = \int_{-\alpha_j}^{\alpha_j} f(x + iy)e^{-2\pi i tx} \, dx.$$

Thus far, we’ve established that $\int_{\alpha_j} f(z)e^{-2\pi i tz} \, dz = 0$ and that the integral along both vertical edges of $\Gamma_{\alpha_j}$ converges to zero as $j \to \infty$. Therefore, we deduce that

$$\lim_{j \to \infty} \left( e^{2\pi ty} g_j(y, t) - e^{2\pi t} g_j(1, t) \right) = 0. \quad (4.6)$$

For sake of notation, we will write $f_y(x) = f(x + iy)$. Then by (4.3) we have $f_y \in L^2(\mathbb{R})$ and then by Plancherel we see that

$$\lim_{j \to \infty} \int_\mathbb{R} |\hat{f}_y(t) - g_j(y, t)|^2 \, dt = 0.$$

This means that $g_j(y, t)$ converges to $\hat{f}_y(t)$ a.e. So if we set $F(t) = e^{2\pi t} \hat{f}_1(t)$ then by (4.6) we also have $F(t) = e^{2\pi ty} f_y(t)$ for any $0 < y < \infty$.

We can use Plancherel again to deduce that

$$\int_\mathbb{R} e^{-4\pi ty}|F(t)|^2 \, dt = \int_\mathbb{R} |\hat{f}_y(t)|^2 \, dt = \int_\mathbb{R} |f_y(x)|^2 \, dx \leq C.$$
If we send \( y \to \infty \) then the above result necessarily gives \( F(t) = 0 \) a.e. in \((-\infty, 0)\). If we send \( y \to 0 \), then the inequality gives \( \int_{\mathbb{R}} |F(t)|^2 \, dt \leq C \). Now since \( F(t) = e^{2\pi yt} \hat{f}_y(t) \) for any \( 0 < y < \infty \) this means that

\[
\int_{\mathbb{R}} |\hat{f}_y(t)| \, dt = \int_{\mathbb{R}} |e^{-2\pi ty} F(t)| \, dt \leq \|e^{-2\pi ty}\|_{L^2(\mathbb{R})} \|F\|_{L^2(\mathbb{R})} < \infty.
\]

That is, \( \hat{f}_y(t) \in L^1(\mathbb{R}) \). Therefore we can write \( f(y)(x) = \int_{\mathbb{R}} \hat{f}_y(t) e^{-2\pi i y t} dt \) and hence

\[
f(z) = f(x + iy) = \int_{0}^{\infty} F(t) e^{-2\pi y t} e^{2\pi i z t} \, dt = \int_{0}^{\infty} F(t) e^{2\pi i z t} \, dt
\]

for any \( z \in \Pi^+ \).

This establishes (4.3) and (4.4). Equation (4.5) follows from the Plancherel calculation done at the beginning of this section.

Now to prove the converse to statement (b):

**Theorem 4.3 (Paley-Wiener).** Suppose, \( A, C > 0 \) and \( f \) is an entire function satisfying

\[
|f(z)| \leq Ce^{A|z|}
\]

for all \( z \in \mathbb{C} \) and

\[
\int_{\mathbb{R}} |f(x)|^2 \, dx < \infty.
\]

Then there exists an \( F \in L^2((-A,A)) \) such that

\[
f(z) = \int_{-A}^{A} F(t) e^{2\pi i z t} \, dt, \quad \forall z \in \mathbb{C}.
\]

**Proof.** Fix \( \epsilon > 0 \) and for notation, set \( f_\epsilon(x) = f(x)e^{-\epsilon|x|} \). We claim that

\[
\lim_{\epsilon \to 0} \int_{\mathbb{R}} f_\epsilon(x) e^{-2\pi i t x} \, dx = 0, \quad \text{for } |t| > A.
\]

By Lebesgue dominated convergence, one can verify that \( \lim_{\epsilon \to 0} \|f_\epsilon - f\|_{L^2(\mathbb{R})} = 0 \). Then Plancherel implies that the Fourier transform of \( f_\epsilon \) converges in \( L^2 \) to the Fourier transform, \( F \), of \( f \) restricted to the real axis. Therefore, if true, (4.10) will imply that \( F = 0 \) is supported on \([-A, A]\) and by the Fourier inversion formula, (4.9) will hold for almost every \( z \) on the real axis. Since (4.9) is entire, then it must hold on all of \( \mathbb{C} \) (that is, if \( g \) was another such function satisfying (4.9) then the entire function \( f - g \) would accumulate zeros on the real axis making \( f - g \equiv 0 \) on \( \mathbb{C} \)). This would prove the theorem. Thus we just have to verify claim (4.10).

For each real \( \alpha \), let \( \Gamma_\alpha \) be the path defined by \( \Gamma_\alpha(s) = se^{i\alpha} \) for \( 0 \leq s < \infty \). Set \( \Pi_\alpha = \{ w \in \mathbb{C} : \text{Re}(we^{i\alpha}) > A \} \). Then for \( w \in \Pi_\alpha \) define

\[
\Phi_\alpha(w) = \int_{\Gamma_\alpha} f(z) e^{-wz} \, dz = e^{i\alpha} \int_{0}^{\infty} f(se^{i\alpha}) \exp(-wse^{i\alpha}) \, ds.
\]
Recall that $|f| \leq Ce^{A|z|}$ and so the above integrand is bounded above in modulus by $Ce^{A|e^{we^{itα}}|} = Ce^{[A−Re(we^{itα})]}$. Then by the same methods in to show that (4.2) was analytic, we can say that $\Phi_α$ is analytic on $\Pi_α$.

Consider the special cases $α = 0$ and $α = π$. Then $\Phi_0(w) = \int_0^∞ f(x)e^{-wx} dx$ for $Re(w) > 0$ and $\Phi_π(w) = -\int_0^∞ f(x)e^{-wx} dx$ for $Re(w) < 0$. This is useful because now we can write (4.10), the expression we need to verify, as:

\[
\int_∞^{−∞} f_t(x)e^{−2πitx} dx = \Phi_0(\epsilon + 2πit) - \Phi_π(−\epsilon + 2πit) \quad \forall t \in \mathbb{R}.
\]

Therefore, we need to prove that the right-hand side of (4.11) tends to zero as $\epsilon \to 0$ for all $|t| > A$.

We make the following subclaim: For any $α, β \in \mathbb{R}$ then $\Phi_α$ and $\Phi_β$ agree on $\Pi_α \cap \Pi_β$; that is, they are analytic continuations of each other.

To see this, suppose $0 < β - α < π$. For sake of notation let $γ = \frac{α + β}{2}$ and $η = \cos\left(\frac{β - α}{2}\right)$. If $w = |w|e^{−iγ}$ then a simple calculation reveals

\[
Re(we^{itα}) = Re(|w|e^{i\frac{α+β}{2}}) = |w|η = Re(we^{iβ}).
\]

Therefore $w \in \Pi_α \cap \Pi_β$ for all $|w| > A/η$ by construction of $\Pi_α$ and $\Pi_β$.

Now examine $\int_{Γ t} f(z)e^{−wz} dz$ were $Γ = Γ(t) = re^{it}$ for $α \leq t \leq β$. Since $Re(−wz) = −|w|r \cos(t − γ) ≤ −|w|rη$ then we can estimate the modulus of the integrand $|f(z)e^{−wz}| ≤ Ce^{r(A−|w|η)}$. This shows that the integral $\int_{Γ t} f(z)e^{−wz} dz \to 0$ as $r \to ∞$. Therefore by Cauchy’s Theorem, $\int_{Γ α} f(z)e^{−wz} dz = \int_{Γ β} f(z)e^{−wz} dz$; that is, $Φ_α(w) = Φ_β(w)$ for all $w = |w|e^{−iγ}$ with $|w| > A/η$. All such points $w$ certainly accumulate therefore we can conclude that the functions are the same; that is $Φ_α = Φ_β$ on $\Pi_α \cap \Pi_β$. This proves the subclaim.

The subclaim is useful because now (4.11) is equivalent to:

\[
\int_∞^{−∞} f_t(x)e^{−2πitx} dx = \Phi_{π/2}(\epsilon + 2πit) - \Phi_{−π/2}(−\epsilon + 2πit) \quad \forall t < −A,
\]

and

\[
\int_∞^{−∞} f_t(x)e^{−2πitx} dx = \Phi_{−π/2}(\epsilon + 2πit) - \Phi_{−π/2}(−\epsilon + 2πit) \quad \forall t > A.
\]

Figure 2: The half-plane region $\Pi_α$ which is essentially the standard right-half plane rotated by $−α$ radians and translated $A$ units to the right.

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Since $\Phi_\alpha$ is analytic then these each certainly go to zero as $\epsilon \to 0$. This proves the original claim (4.2), which proves the theorem.

There is another result that bears the name Paley-Wiener Theorem which we state here:

**Theorem 4.4 (Paley-Wiener).** For $f \in L^2(\mathbb{R})$, the following two conditions are equivalent:

1. $f$ is the restriction to $\mathbb{R}$ of a function $F$ analytic in the strip $\{z \in \mathbb{C} : |\Im(z)| < a\}$ and satisfying
   \begin{equation}
   \int |F(x + iy)|^2 \, dx \leq c
   \end{equation}
   for some constant $c$, for all $|y| < a$.

2. \begin{equation}
   e^{2\pi a|\xi|} \hat{f} \in L^2(\mathbb{R}).
   \end{equation}

**Proof.** ($2 \Rightarrow 1$): Suppose $e^{2\pi a|\xi|} \hat{f} \in L^2(\mathbb{R})$. Write

\begin{equation}
F(z) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi z} \, d\xi.
\end{equation}

Then by the Fourier inversion formula, we have $F|_\mathbb{R} = f$. For the inversion to be well-defined, by (4.13), we can only consider $z \in \mathbb{C}$ such that $\{|\Im(z)| < a\}$. By writing $e^{2\pi i \xi z}$ as a power series in $z$ and integrating each term, we see that $F$ is analytic in $\{|\Im(z)| < a\}$.

Then by Plancherel:

\begin{align*}
\int_{-\infty}^{\infty} |F(x + iy)|^2 \, dx &= \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 e^{4\pi^2 y^2} \, d\xi \\
&= \int_{-\infty}^{0} |\hat{f}(\xi)|^2 e^{4\pi^2 y^2} \, d\xi + \int_{0}^{\infty} |\hat{f}(\xi)|^2 e^{4\pi^2 y^2} \, d\xi \\
&= \int_{-\infty}^{0} |\hat{f}(\xi)|^2 e^{4\pi^2 (-y)^2} \, d\xi + \int_{0}^{\infty} |\hat{f}(\xi)|^2 e^{4\pi^2 y^2} \, d\xi \\
&\leq \left\| e^{2\pi a|\xi|} \hat{f} \right\|_{L^2(\mathbb{R})}^2.
\end{align*}

($1 \Rightarrow 2$): Now suppose $F$ is analytic in the strip $\{z \in \mathbb{C} : |\Im(z)| < a\}$ and satisfies $\int |F(x + iy)|^2 \, dx \leq c$. Write $f_y(x) = F(x + iy)$ (therefore $f = f_0$) and consider their Fourier transforms $\hat{f}_y$. We claim that $\hat{f}_y(\xi) = \int \hat{f}(\xi) e^{-2\pi i \xi y} \, d\xi$ because then by Plancherel and (4.12), we would have $\int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 e^{4\pi^2 y^2} \, d\xi < c$ for all $|y| < a$ which implies (4.13).

To prove the claim, let $\lambda > 0$ and $z$ in the strip $\{|\Im(z)| < a\}$. Let

\begin{equation}
G_\lambda(z) = K_\lambda * F = \int_{\mathbb{R}} F(z - u)K_\lambda(u) \, du.
\end{equation}
Notice $G_\lambda$ is analytic in the strip $\{ |\Im(z)| < a \}$. Moreover, if we denote $g_{\lambda,y}(x) = G_\lambda(x + iy) = K_\lambda * f_y$ then $\hat{g}_{\lambda,y}(\xi) = \hat{K}_\lambda(\xi) \hat{f}_y(\xi)$. By the properties of $\hat{K}_\lambda$, we know that $\hat{g}_{\lambda,y}(\xi)$ has compact support in $[-\lambda, \lambda]$. Then

$$\hat{g}_{\lambda,y}(\xi) = \hat{K}_\lambda(\xi) \hat{f}_y(\xi) = \hat{K}_\lambda(\xi) \hat{f}_0(\xi) e^{-2\pi i \xi (-iy)} = \hat{K}_\lambda(\xi) \hat{f}_0(\xi) e^{-2\pi i \xi y} = \hat{g}_{\lambda,0}(\xi) e^{-2\pi i \xi y}.$$

Consequently, if $|\xi| < \lambda$, then $\hat{f}_y(\xi) = \hat{f}(\xi) e^{-2\pi i \xi y}$. But since $\lambda > 0$ was arbitrary, the above result holds for any $\xi \in \mathbb{R}$.
5 Convergence of Fourier Series

We return to the setting of the torus $\mathbb{T}$ and look to answer the question of when the Fourier series $S_n(f)$ converges to $f$.

5.1 Recap and review

Recall the definition of a homogeneous Banach space.

**Definition 5.1.** A *homogeneous Banach space* on $\mathbb{T}$ is a linear subspace $B$ of $L^1(\mathbb{T})$ that is Banach under the norm $\| \cdot \|_B \geq \| \cdot \|_{L^1(\mathbb{T})}$ with the following two properties:

1. \[ \text{If } f \in B \text{ and } a \in \mathbb{T}, \text{ then } \tau_a f(t) = f(t-a) \in B \text{ and } \| \tau_a f \|_B = \| f \|_B. \]
2. \[ \text{For all } f, a, a_0 \in \mathbb{T} \text{ then } \lim_{a \to a_0} \| \tau_a f - \tau_{a_0} f \|_B = 0. \]

An example of a homogeneous Banach space is $C(\mathbb{T})$ with the norm $\| f \|_{L^\infty(\mathbb{T})}$. Another is $L^p(\mathbb{T})$ for $1 \leq p < \infty$ with the norm $\| f \|_{L^p(\mathbb{T})} = (\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p \, dt)^{1/p}$.

Recall that the Féjer kernel is denoted $K_n(t) = \sum_{|j| \leq n} \left(1 - \frac{|j|}{n+1}\right) e^{2\pi ijt}$ and we have $\|K_n\|_{L^\infty(\mathbb{T})} = 1$ for any $n \geq 0$. For any $f \in L^1(\mathbb{T})$, we use the notation $\sigma_n(f) = K_n * f = \sum_{j=-n}^{n} \left(1 - \frac{|j|}{n+1}\right) \hat{f}(j)e^{2\pi ijt}$.

Recall that $S_n(f)$ denotes the symmetric partial sum of the Fourier series of $f$, that is, $S_n(f) = \sum_{j=-n}^{n} \hat{f}(j)e^{2\pi ijt}$. We also denote by $S_n$, the operator which sends $f \in B$ to $S_n(f)$. The norm of this operator is denoted $\|S_n\|_B$.

5.2 Convergence in Norm

**Definition 5.2.** We say that the homogeneous Banach space, $B$ *admits convergence in norm* if

\[ \lim_{n \to \infty} \|S_n(f) - f \|_B = 0. \]

**Theorem 5.3.** A homogeneous Banach space, $B$, admits convergence in norm iff $\|S_n\|^B$ is bounded as $n \to \infty$, that is, if there exists some $K > 0$ such that

\[ \|S_n(f)\|_B \leq K \|f\|_B \]

for all $f \in B$ and for all $n \geq 0$.

*Proof.* ($\Rightarrow$): Suppose $B$ admits convergence in norm. Then $S_n(f)$ converges to $f$ for all $f \in B$. Therefore, $S_n(f)$ is bounded almost everywhere on $\mathbb{T}$ for every $f \in B$. Then by the uniform boundedness principle, $\sup_n \|S_n\|^B < \infty$, which is precisely what was desired.

($\Leftarrow$): Now suppose that $\|S_n\|^B < K$ for all $n$. Without loss of generality, we may assume that $K \geq 1$. Let $f \in B$ and fix $\epsilon > 0$. Let $P$ be a trigonometric polynomial such that $\|f - P\|_B < \frac{\epsilon K}{2\pi}$ which is guaranteed by density of trigonometric polynomials in $B$ (note that we return to the usual definition of trigonometric polynomials, that is, any $P(t) = \sum_{n=-N}^{N} a_n e^{2\pi int}$, as compared
to trigonometric polynomials used for almost-periodic functions). Then for any \( n \geq \deg(P) \) we have \( S_n(P) = P \) and so
\[
\|S_n(f) - f\|_B = \|S_n(f) - S_n(P) + P - f\|_B \\
\leq \|S_n(f - P)\|_B + \|P - f\|_B < \frac{K\epsilon}{2K} + \frac{\epsilon}{2K} \leq \epsilon
\]
which proves the result.

**Example 5.4.** Recall that \( S_n(f) = D_n \ast f \) where \( D_n \) is the Dirichlet kernel \( D_n(t) = \sum_{j=-n}^{n} e^{2\pi ijt} \). Then \( \|S_n(f)\|_B = \|D_n \ast f\|_B \leq \|D_n\|_{L^1(\mathbb{T})} \|f\|_B \). Hence, we obtain an upper estimate for the norm of the operator \( S_n \), which we call **Lebesgue constants** denoted
\[ L_n = \|D_n\|_{L^1(\mathbb{T})} \geq \|S_n\|^B. \]

It can be show that the Lebesgue constants tend to infinity like \( \log(n) \).

In the case \( B = L^1(\mathbb{T}) \) we see that the norm of \( S_n \) equals the Lebesgue constants. To see this, recall that
\[
\hat{K}_n(j) = \begin{cases} 
  1 - \frac{|j|}{n + 1}, & \text{for } |j| \leq n + 1 \\
  0, & \text{otherwise}
\end{cases}, \quad \text{and } \hat{D}_n(j) = \begin{cases} 
  1, & \text{for } |j| \leq n \\
  0, & \text{otherwise}.
\end{cases}
\]

Then
\[
\|S_n\|^{L^1(\mathbb{T})} \geq \|S_n(K_N)\|_{L^1(\mathbb{T})} = \left\| \sum_{j=-n}^{n} \left( 1 - \frac{|j|}{N + 1} \right) e^{2\pi ijt} \right\|_{L^1(\mathbb{T})} = \|\sigma_N(D_n)\|_{L^1(\mathbb{T})}.
\]

We’ve seen that \( \lim_{N \to \infty} \sigma_N(D_n) = D_n \) and so \( \|S_n\|^{L^1(\mathbb{T})} \geq L_n \); and hence
\[
\|S_n\|^{L^1(\mathbb{T})} = L_n = \|D_n\|_{L^1(\mathbb{T})}.
\]

Therefore, \( L^1(\mathbb{T}) \) does not admit convergence in norm.

**Example 5.5.** In the case \( B = C(\mathbb{T}) \), convergence in norm is equivalent to uniform convergence. Fix \( \epsilon > 0 \) and consider the continuous functions \( \psi_n \) such that \( \|\psi_n\|_{L^\infty(\mathbb{T})} \leq 1 \) and such that \( \phi_n(t) = \text{sgn}(D_n(t)) \) except in small intervals around the points of discontinuity of \( \text{sgn}(D_n(t)) \). By Lusin’s Theorem, we can make the sum of these lengths less than \( \epsilon/2n \). Then
\[
\|S_n\|^{C(\mathbb{T})} \geq \|S_n(\psi_n)\|_{L^\infty(\mathbb{T})} \geq |S_n(\psi, 0)| = |D_n \ast \psi(0)| \leq \int_{\mathbb{T}} \left| D_n(t) \psi_n(t) \right| dt > L_n - \epsilon.
\]

Since \( \epsilon > 0 \) was arbitrary and since \( L_n \) grows logarithmic, this shows that \( \|S_n\|^{C(\mathbb{T})} \) is unbounded. Hence, \( C(\mathbb{T}) \) does not admit convergence in norm.

For homogeneous Banach spaces on \( \mathbb{T} \), the problem of convergence in norm can be related to invariance under conjugation.
Theorem 5.9. Let \( f \) be a trigonometric polynomial \( \sum_n a_n e^{2\pi i n t} \), we define the conjugate series to be \( -i \sum_n \text{sgn}(n) a_n e^{2\pi i n t} \). If \( f \in L^1(\mathbb{T}) \) and if the conjugate to \( \sum_n \hat{f}(n) e^{2\pi i n t} \) is the Fourier series of some function \( g \in L^1(\mathbb{T}) \), then we say \( g \) is the conjugate function of \( f \), denoted \( \hat{f} \).

Definition 5.7. A space of functions \( B \subset L^1(\mathbb{T}) \) admits conjugation if for every \( f \in B \), \( \hat{f} \) is defined and belongs to \( B \).

If \( B \) is a homogeneous Banach space which admits conjugation, then the mapping \( f \rightarrow \hat{f} \) is a bounded linear operator on \( B \). The linearity is straightforward to show. The boundedness relies on the closed graph theorem; that is, if \( \lim_{n \to \infty} \hat{f}_n = f \) and \( \lim_{n \to \infty} \hat{f}_n = g \) in \( B \), then we need to show that \( g = \hat{f} \).

Fix any integer \( j \), then
\[
\hat{g}(j) = \lim_{n \to \infty} \hat{f}_n(j) = -i \text{sgn}(j) \lim_{n \to \infty} \hat{f}_n(j) = -i \text{sgn}(j) \hat{f}(j) = \hat{f}(j).
\]
and by the uniqueness theorem, \( g = \hat{f} \). So, the homogeneous Banach space, \( B \), admits conjugation.

Remark 5.8. We will utilize the following facts to prove the upcoming theorem: If \( B \) admits conjugation, then the mapping \( f \rightarrow f^* = \frac{1}{2} \hat{f}(0) + \frac{1}{2}(f + i\hat{f}) \sim \sum_{j=0}^{\infty} \hat{f}(j)e^{2\pi ij t} \) is a well-defined, bounded linear map on \( B \). Conversely, if the mapping \( f \rightarrow f^* \) is well-defined on a space \( B \), then \( B \) admits conjugation since \( \hat{f} = -i(2f^* - f - \hat{f}(0)) \).

Theorem 5.9. Let \( B \) be a homogeneous Banach space on \( \mathbb{T} \) and assume that for \( f \in B \) and for any \( n \),
\[
\|e^{2\pi i n t} f\|_B = \|f\|_B.
\]
Then \( B \) admits conjugation iff \( B \) admits convergence in norm.

**Proof.** By Theorem 5.3 and the foregoing remarks, we have to show that the mapping \( f \rightarrow f^* \) is well-defined in \( B \) iff the operators \( S_n \) are uniformly bounded on \( B \).

\((\Leftarrow)\): Assume first that \( \|S_n\| \leq K \) for some constant \( K > 0 \) for all \( n \geq 0 \).

We shall define
\[
S_n^0(f) = \sum_{j=0}^{2n} \hat{f}(j)e^{2\pi ij t} = e^{2\pi i n t} S_n(e^{-2\pi i n t} f).
\]

Then by using (5.2) we obtain
\[
\left\|S_n^0(f)\right\|_B = \left\|e^{2\pi i n t} S_n(e^{-2\pi i n t} f)\right\|_B = \left\|S_n(e^{-2\pi i n t} f)\right\|_B \leq \|S_n\| \|e^{-2\pi i n t} f\|_B \leq K \|f\|_B
\]
which implies \( \|S_n^0\| \leq K \) for all \( n \).

Now let \( f \in B \) \( \epsilon > 0 \). Furthermore, let \( P \in B \) be a trigonometric polynomial such that \( \|f - P\|_B \leq \epsilon/2K \). Then
\[
\left\|S_n^0(f) - S_n^0(P)\right\|_B = \left\|S_n^0(f - P)\right\|_B \leq \epsilon/2.
\]
As long as $n, m > \deg(P)$ then $S^*_{n}(P) = S^*_{m}(P)$. Then by the triangle inequality, we have $\|S^*_{n}(f) - S^*_{m}(f)\|_{B} \leq \epsilon$. Now $\{S^*_{n}(f)\}_{n=1}^{\infty}$ is a Cauchy sequence in $B$; hence it converges and its limit has the Fourier series $\sum_{j}^{\infty} \hat{f}(j)e^{2\pi ij t}$. Hence $f^{b} = \lim_{n \to \infty} S^*_{n}(f) \in B$ is well-defined and thus $B$ admits conjugation.

$(\Rightarrow)$: Now suppose that the operator $f \to f^{b}$ is well-defined in $B$ (we get boundedness for free by the remarks). Then $S^*_{n}(f) = f^{b} - e^{2\pi i (2n+1)t}(e^{-2\pi i (2n+1)t} f)^{b}$.

By the triangle inequality and (5.2) imply that $\|S^*_{n}(\cdot)\|_{B}$ is bounded above by twice the norm over $B$ of the mapping $f \to f^{b}$. By (5.3) and (5.2) we have $\|S_{n}\|_{B} = \|S^*_{n}\|_{B}$ which we’ve just shown is uniformly bounded. Therefore $B$ admits convergence in norm.

5.3 Convergence and Divergence at a Point

In Example 5.5 we saw that for $f \in C(T)$, $S_{n}(f)$ need not converge uniformly on $T$. We now will explore the fact that $S_{n}(f)$ need not even converge pointwise.

**Theorem 5.10.** There exists a continuous function on $T$ whose Fourier series diverges at a point.

**Proof.** We first claim that the mapping $T_{n}(f) = S_{n}(f, 0)$ is a bounded linear functional on $C(T)$. The linearity is straightforward to check. As for the boundedness, notice that for any $f \in C(T)$

$$|S_{n}(f, 0)| = \left| \sum_{j=-n}^{n} \hat{f}(j) \right| = \left| \sum_{j=-n}^{n} \int_{T} f(t)e^{-2\pi ij t} dt \right| \leq (2n + 1) \|f\|_{L^{\infty}(T)}.$$

So indeed the linear map $T_{n}$ is continuous with norm $\|T_{n}\| \leq (2n + 1)$. This family is clearly not uniformly bounded in $n$. Therefore, by the uniform boundedness principle, there exists some $f \in C(T)$ such that $S_{n}(f, 0)$ diverges as $n \to \infty$ (Actually, the uniform boundedness principle says that such functions $f$ are dense in $C(T)$). Thus, we are guaranteed the existence of a continuous function on $T$ whose Fourier series diverges at $t = 0$.

We now give one convergence criterion due to Hardy:

**Theorem 5.11.** Let $f \in L^{1}(T)$ and assume $\hat{f}(n) = O(1/n)$ as $n \to \infty$. Then $S_{n}(f, t)$ and $\sigma_{n}(f, t)$ converge to the same limit for the same values of $t$. Moreover, if $\sigma_{n}(f, t)$ converges uniformly on some subset of $T$, then $S_{n}(f, t)$ also converges uniformly on the same subset.

**Proof.** $\hat{f}(n) = O(1/n)$ implies that for every $\epsilon > 0$, there exists some integer $\lambda > 1$ such that

$$\limsup_{n \to \infty} \sum_{n < |j| \leq \lambda n} |\hat{f}(j)| < \epsilon.$$
Then by algebraic manipulations, we can write
\[
S_n(f,t) = \sum_{j=-n}^{n} \hat{f}(j) e^{2\pi ijt} = \frac{\lambda n + 1}{\lambda n - n} \sigma_n(f,t) - \frac{n + 1}{\lambda n - n} \sigma_n(f,t) - \lambda n + 1 \sum_{n<|j|\leq \lambda n} \left(1 - \frac{|j|}{\lambda n + 1}\right) \hat{f}(j) e^{2\pi ijt}.
\]

By our assumption, there exists some \(n_0\) such that for all \(n > n_0\), the third term above is less than \(\epsilon/2\). Now if there exists some \(t_0 \in \mathbb{T}\) with \(\sigma_n(f,t_0) \rightarrow \sigma(f,t_0)\), then the first two terms above converge to \(\sigma(f,t)\). Therefore, for \(n\) sufficiently large, larger than some \(n_1\), we can assert that \(|S_n(f,t_0) - \sigma(f,t_0)| < \epsilon/2 + \epsilon/2 = \epsilon\). That is, \(\lim_{n \to \infty} S_n(f,t_0) = \sigma(f,t_0)\).

Our choice of \(n_1\) depended only on the convergence of \(\sigma_n(f,t_0) \rightarrow \sigma_n(f,t_0)\). Therefore if \(\sigma_n(f,t) \rightarrow \sigma(f,t)\) uniformly on some subset of \(\mathbb{T}\), then so does \(S_n(f,t)\).

\[\text{Remark 5.12.} \quad \text{To prove the above theorem, we didn’t use the full power of } \hat{f}(n) = O(1/n); \text{we merely used it to assume } \limsup_{n \to \infty} \sum_{n<|j|\leq \lambda n} |\hat{f}(j)| < \epsilon.\]

We could replace \(\hat{f}(n) = O(1/n)\) with this weaker requirement.

\[\text{Corollary 5.13.} \quad \text{Let } f \in BV(\mathbb{T}). \text{ Then } \lim_{n \to \infty} S_n(f,t) = \frac{1}{2} \lim_{h \to 0} (f(t+h) + f(t-h)). \text{ In particular, } \lim_{n \to \infty} S_n(f,t) = f(t) \text{ at every point of continuity of } f. \text{ Moreover, this convergence is uniform on any closed intervals of continuity of } f.\]

\[\text{Proof.} \quad \text{Since } f \in BV(\mathbb{T}), \text{ then the limit } \lim_{h \to 0} (f(t+h) + f(t-h)) \text{ exists. Then by Féjer’s Theorem, } \lim_{n \to \infty} \sigma_n(f,t) = \lim_{h \to 0} (f(t+h) + f(t-h)). \text{ We proved in the previous document (in the section titled “Order of Magnitude of Fourier Coefficients”) that if } f \in BV(\mathbb{T}), \text{ then } |\hat{f}(n)| \leq \frac{\text{Var}(f)}{|n|}. \text{ This allows us to use the previous theorem which proves the corollary.}\]

\[\text{Lemma 5.14.} \quad \text{Let } f \in L^1(\mathbb{T}) \text{ and assume } \int_{-1}^{1} \left|\frac{f(t)}{t}\right| dt < \infty. \text{ Then } \lim_{n \to \infty} S_n(f,0) = 0.\]

\[\text{Proof.} \quad \text{Recall that the Dirichlet kernel } D_n(t) = \sum_{j=-n}^{n} e^{2\pi ijt} = \frac{\sin((n + 1/2)2\pi t)}{\sin(\pi t)}. \text{ Then by using the trigonometric identity } \sin(\alpha + \beta) = \sin(\alpha) \cos(\beta) + \cos(\alpha) \sin(\beta), \text{ we have}\]

\[S_n(f,0) = \int_{\mathbb{T}} f(t) \frac{\sin((n + 1/2)2\pi t)}{\sin(\pi t)} dt = \int_{\mathbb{T}} f(t) \cos(2\pi nt) dt + \int_{\mathbb{T}} f(t) \frac{\cos(\pi t)}{\sin(\pi t)} \sin(2\pi nt) dt.\]

By the assumption, \(f(t) \frac{\cos(\pi t)}{\sin(\pi t)} \in L^1(\mathbb{T})\) (since \(\left|\frac{\cos(\pi t)}{\sin(\pi t)}\right| \leq |1/t|\) on \([-1,1]\)) and so by the Riemann-Lebesgue lemma, all integrals tend to zero.
Theorem 5.15 (Principle of localization). Let \( f \in L^1(\mathbb{T}) \) and assume that \( f \) vanishes in an open interval \( I \). Then \( S_n(f,t) \) converges to zero for \( t \in I \), and the convergence is uniform on closed subintervals of \( I \).

Proof. The convergence to zero at \( t \in I \) is an immediate consequence of the previous lemma (by simply translating \( f \)). If \( I_0 \) is a closed subinterval of \( I \) and \( t_0 \in I \), the functions \( \phi_{t_0}(t) = \frac{f(t-t_0)\cos(\pi t)}{\sin(\pi t)} \) is a compact family in \( L^1(\mathbb{T}) \). To see this, we use the fact that for any \( g \in L^1 \), \( \|T_t g - T_s g\|_{L^1} \to 0 \) as \( t \to s \) and this convergence is uniform in \( s \). Therefore, we can cover the family \( \{\phi_{t_0}(t)\}_{t_0 \in I_0} \) with a finite number of open balls \( \{B(\phi_{t_i}(t), \delta)\}_{i=1}^N \) which shows compactness of \( \{\phi_{t_0}(t)\}_{t_0 \in I_0} \).

We next claim that the Riemann-Lebesgue Lemma holds uniformly on compact subsets of \( L^1(\mathbb{T}) \). This is because if \( K \subseteq L^1(\mathbb{T}) \) is compact and \( \epsilon > 0 \) then we can cover \( K \) with a finite number of open balls centered at trigonometric polynomials \( P_1, ..., P_N \) such that for any \( f \in K \), \( \|f - P_j\|_{L^1(\mathbb{T})} < \epsilon \) for at least one \( 1 \leq j \leq N \). Therefore for \( |n| > \max_{1 \leq j \leq N} \deg(P_j) \), we have

\[
|\hat{f}(n)| = |(\hat{f} - \hat{P})(n)| \leq \|f - P\|_{L^1(\mathbb{T})} < \epsilon
\]

and thus the Riemann-Lebesgue lemma holds uniformly on \( K \).

This now implies that the integrals in the proof of Lemma 5.14 corresponding to \( f(t-t_0) \) with \( t_0 \in I_0 \) tend to zero uniformly. \( \square \)

An immediate corollary of the previous two results is:

Corollary 5.16 (Dini’s Test). Let \( f \in L^1(\mathbb{T}) \). If

\[
\int_{-1}^{1} \left| \frac{f(t-t_0) - f(t_0)}{t} \right| dt < \infty
\]

then \( S_n(f,t_0) \to f(t_0) \).

References

