THE UNCERTAINTY PRINCIPLE: A BRIEF SURVEY

MATTHEW BEGUÉ

CONTENTS

1. Introduction 1
2. $L^2(\mathbb{R})$ - Heisenberg’s Inequality 1
3. Hilbert spaces 3
4. Variations on Heisenberg’s Inequality - $L^p$ spaces 5
5. Local Uncertainty Inequalities 6
6. Other generalizations, variations, modifications 7
References 8

1. Introduction

The uncertainty principle is a cornerstone in quantum physics. However, its principles play an equally monumental role in harmonic analysis. To put it in one sentence: A nonzero function and its Fourier transform cannot both be sharply localized.

While Heisenberg gave a clear physical interpretation of the uncertainty principal in 1927 in [7], it contains little mathematical precision. This was remedied later by Kennard ([8]) and Weyl ([11]) in 1928. We still credit the famous inequality (2.1) to Heisenberg.

The paper is structured as follows: In Section 2, we present the simplest and most common form of the uncertainty principle in harmonic analysis (Heisenberg’s inequality). We then extend Heisenberg’s inequality to Hilbert spaces (Section 3) and $L^p$ spaces (Section 4). In Section 5, we present some of the results from local uncertainty inequalities and their ramifications. We end the paper (Section 6) with a grab-bag of other interesting results.

Uncertainty principle results and inequalities are bountiful; this paper does not even scratch the surface of the hundreds of results on this topic. The information in this document primarily comes from Folland’s “The Uncertainty Principle: A Mathematical Survey” [6] which is why this document is titled “... : A Brief Survey”. Nonetheless, we present many of the important uncertainty results with full proofs, sketches of proofs, and, otherwise, references for the reader.

2. $L^2(\mathbb{R})$ - Heisenberg’s Inequality

The simplest formulation of the uncertainty principle in harmonic analysis is Heisenberg’s inequality. This gives an uncertainty principal for functions in $L^2(\mathbb{R})$. We will first prove Heisenberg’s inequality in the Schwartz space $\mathcal{S}(\mathbb{R})$ because of the simplicity and beauty of the proof.
Theorem 2.1 (Heisenberg’s Inequality). For \( f \in \mathcal{S}(\mathbb{R}) \), then for any \( a, b \in \mathbb{R} \)

\[
(2.1) \quad \left( \int_{\mathbb{R}} (x-a)^2 |f(x)|^2 \, dx \right) \left( \int_{\mathbb{R}} (\xi-b)^2 |\hat{f}(\xi)|^2 \, d\xi \right) \geq \frac{\|f\|_{L^2(\mathbb{R})}^4}{16\pi^2}
\]

Moreover, equality holds if and only if \( f(x) = Ce^{2\pi i bx}e^{-\gamma(x-a)^2} \) for some \( C \in \mathbb{R} \) and \( \gamma > 0 \).

Proof. Assume first (without loss of generality) that \( a = b = 0 \). We can assert that the first integral in (2.1) is finite since \( f \in \mathcal{S}(\mathbb{R}) \). For any differentiable function,

\[
(2.2) \quad \hat{(f')}(\xi) = 2\pi i \xi \hat{f}(\xi)
\]

(\( f' \) exists since we’ve assumed \( f \in \mathcal{S}(\mathbb{R}) \)). By Plancherel, \( \|\hat{(f')}\|_{L^2(\mathbb{R})}^2 = \|f'\|_{L^2(\mathbb{R})}^2 \), we can assert that the second integral of (2.1) is finite.

A simple calculation reveals

\[
\frac{d}{dx}|f(x)|^2 = \frac{d}{dx}f(x)f'(x) = f(x)f'' + f'(\bar{f}) + f(\bar{f}') + f\bar{f}' = 2\text{Re} f\bar{f}'
\]

and so by integration by parts, for \(-\infty < c < d < \infty\) we have

\[
2\text{Re} \int_c^d x f(x) f'(x) \, dx = x|f(x)|^2|_c^d - \int_c^d |f(x)|^2 \, dx.
\]

We know \( f(x), f'(x), xf(x) \in L^2(\mathbb{R}) \) (since \( f \in \mathcal{S}(\mathbb{R}) \)) which implies the boundary terms vanish as \( c \to -\infty, d \to +\infty \). So we have

\[
\int_{\mathbb{R}} |f(x)|^2 \, dx = -2\text{Re} \int_{\mathbb{R}} xf(x) \bar{f}'(x) \, dx.
\]

Now by Hölder’s inequality, Plancharel, and (2.2), we can compute:

\[
\|f\|_{L^2(\mathbb{R})}^4 = \left( \int_{\mathbb{R}} |f|^2 \, dx \right)^2 \left( -2\text{Re} \int_{\mathbb{R}} xf(x) \bar{f}'(x) \, dx \right)^2
\]

\[
\leq 4 \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \int_{\mathbb{R}} |f'(x)|^2 \, dx
\]

\[
= 4 \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \int_{\mathbb{R}} |\hat{f}'(\xi)|^2 \, d\xi
\]

\[
= 4 \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \int_{\mathbb{R}} 2\pi i \xi \hat{f}(\xi)^2 \, d\xi
\]

\[
= 16\pi^2 \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi
\]

which gives the desired result.

To show when (2.1) holds with equality, recall that the Cauchy-Schwarz (Hölder) inequality holds with equality if and only if the two functions are linearly dependent (i.e. scalar multiples of each other). Therefore, (2.1) holds with equality if and only if \( f'(x) = Cxf(x) \) for \( C \in \mathbb{R} \). For notational purposes, we can equivalently write this as \( f'(x) = -2\gamma xf(x) \) which has solution \( f(x) = C'e^{-\gamma x^2} \) with \( \gamma > 0 \) necessary to preserve \( f \in L^2(\mathbb{R}) \).

Finally, we address why the assumption \( a = b = 0 \) was without loss of generality. We have just proved that \( \left( \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \right) \left( \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi \right) \geq \frac{\|f\|_{L^2(\mathbb{R})}^4}{16\pi^2} \). Now replace \( f(x) \) with
\(e^{2\pi i x b} f(x + a)\). Elementary calculations reveal

\[
\left( \int_{\mathbb{R}} x^2 |e^{2\pi i x b} f(x + a)|^2 \, dx \right) = \left( \int_{\mathbb{R}} x^2 |f(x + a)|^2 \, dx \right) = \left( \int_{\mathbb{R}} (x - a)^2 |f(x)|^2 \, dx \right),
\]

\[
\left( \int_{\mathbb{R}} \xi^2 |(e^{2\pi i x b} f(x - a))^\wedge|^2 \, d\xi \right) = \left( \int_{\mathbb{R}} \xi^2 |e^{2\pi i a(\xi + b)} \hat{f}(\xi + b)|^2 \, d\xi \right) = \left( \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi + b)|^2 \, d\xi \right) = \left( \int_{\hat{\mathbb{R}}} (\xi - b)^2 |\hat{f}(\xi)|^2 \, d\xi \right),
\]

and

\[
\|e^{-2\pi i x b} f(x + a)\|_{L^2(\mathbb{R})}^4 = \|f\|_{L^2(\mathbb{R})}^4.
\]

So the generalized inequality (2.1) holds too. \(\square\)

The Heisenberg’s Inequality that we’ve stated and proved is for functions in the Schwartz space. The result is valid for functions in \(L^2(\mathbb{R})\). This is done by a closure argument and is given in full detail in [1].

**Remark 2.2.** There is a probabilistic interpretation to Theorem 2.1. Let \(f \in L^2(\mathbb{R})\) and suppose \(\|f\|_{L^2(\mathbb{R})} = 1\). Then \(f\) can be thought of as a probability density function. We define the variance of \(f\) to be

\[
V(|f|^2) = \inf_{a \in \mathbb{R}} \int_{\mathbb{R}} (x - a)^2 |f(x)|^2 \, dx
\]

and similarly the variance of \(\hat{f}\) to be

\[
V(|\hat{f}|^2) = \inf_{b \in \mathbb{R}} \int_{\mathbb{R}} (\xi - b)^2 |\hat{f}(\xi)|^2 \, d\xi.
\]

Then, Heisenberg’s inequality states that the product of the variances of \(f\) and \(\hat{f}\) are bounded below by \(\frac{1}{16\pi^2}\). That is,

\[
V(|f|^2)V(|\hat{f}|^2) \geq \frac{1}{16\pi^2}.
\]

**Remark 2.3.** It is worth mentioning that there is nothing significant about the constant \(16\pi^2\). This is an artifact of which definition of the Fourier transform one uses. We are using \(\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi i x \xi} \, dx\). Different definitions of \(f(x)\) yield different constants, but the principle is sound.

3. **Hilbert spaces**

One of the nicest properties of \(L^2(\mathbb{R})\) is that it is the only \(L^p\) space that is a Hilbert space, that is, a Banach space equipped with an inner product. It turns out that we can generalize an uncertainty principle to general Hilbert spaces. We first require a few definitions:
We compute the right-hand side by using the definition of the expectation and combine it with Theorem 3.2 which proves the theorem.

\[ 4\Delta_x^2(A)\Delta_x^2(B) \geq [E_x(i[A,B])]^2. \]

**Proof.** We compute the right-hand side by using the definition of the expectation and commutator, as well as exploiting the linearity of the inner product and the self-adjointness of \( A \) and \( B \):

\[
E_x(i[A,B]) = i((A - BA)x, x) = i \left( \langle Ax, Ax \rangle - \langle Ax, Bx \rangle \right)
\]

\[
= i \left( \langle Ax, Bx \rangle - \langle Ax, Bx \rangle \right) = 2 \text{Im} \langle Ax, Bx \rangle.
\]

It should be noted that these calculations are legitimate since \( D(A^2) \subseteq D(A) \) and similarly for \( B \).

Notice next that we must have \( \langle Ax, x \rangle \in \mathbb{R} \) since \( A \) is self-adjoint; that is,

\[ \langle Ax, x \rangle = \langle x, Ax \rangle = \langle Ax, x \rangle \]

and similarly for \( \langle Bx, x \rangle \). As a result, this gives

\[ |\langle (B + iA)x, x \rangle|^2 = |\langle Bx, x \rangle + i\langle Ax, x \rangle|^2 = \langle Bx, x \rangle^2 + \langle Ax, x \rangle^2. \]

By the Cauchy-Schwarz inequality

\[ |\langle (B + iA)x, x \rangle|^2 \leq \|x\|^2 \|B + iA\|^2 \leq \|B + iA\|^2. \]

Some elementary manipulations give:

\[ \| (B + iA)x \|^2 = \langle (B + iA)x, (B + iA)x \rangle \]

\[ = \langle Bx, Bx \rangle - i\langle Bx, Ax \rangle + i\langle Ax, Bx \rangle + i(-i)\langle Ax, Ax \rangle \]

\[ = \| Bx \|^2 + \| Ax \|^2 + i \left( \langle Ax, Bx \rangle - \langle Ax, Bx \rangle \right) \]

\[ = \| Bx \|^2 + \| Ax \|^2 - 2 \text{Im} \langle Ax, Bx \rangle. \]

Combining (3.1), (3.2), and (3.3) gives:

\[ \| Bx \|^2 - \langle Ax, x \rangle^2 + \| Ax \|^2 - \langle Bx, x \rangle^2 \geq 2 \text{Im} \langle Ax, Bx \rangle. \]

Scalar multiplication preserves self-adjointness since \( \langle rAx, x \rangle = r \langle Ax, x \rangle = \overline{r} \langle x, Ax \rangle = \langle x, rAx \rangle \). So rewriting (3.4) with \( rA \) and \( sB \), for \( r, s \in \mathbb{R} \), gives

\[ r^2(\| Ax \|^2 - \langle Ax, x \rangle^2) + s^2(\| Bx \|^2 - \langle Bx, x \rangle^2) \geq 2rs \text{Im} \langle Ax, Bx \rangle. \]

If we let \( r^2 = \| Bx \|^2 - \langle Bx, x \rangle^2 \) and \( s^2 = \| Ax \|^2 - \langle Ax, x \rangle^2 \) then the above inequality becomes

\[ rs \geq \text{Im} \langle Ax, Bx \rangle. \]

Finally, squaring both sides gives

\[ (\| Ax \|^2 - \langle Ax, x \rangle^2)(\| Bx \|^2 - \langle Bx, x \rangle^2) \geq \left( \frac{\text{Im} \langle Ax, Bx \rangle}{2} \right)^2 \]

which proves the theorem. \( \square \)
This generalization of the uncertainty principle to Hilbert spaces is not as insightful as one might think. Firstly, the assumption \( x \in D(A^2) \cap D(B^2) \cap D(i[A, B]) \) is incredibly strong. \( D([A, B]) \) need not be dense in \( H \). In fact, it is possible that \( D([A, B]) = 0 \) in which case theorem provides no information. Another problem is that even if \( D([A, B]) \) was dense in \( H \), the operator \([A, B]\) is usually not a closed operator which destroys the possibility of using any approximation argument.

4. Variations on Heisenberg’s Inequality - \( L^p \) spaces

We now turn to generalize and modify Heisenberg’s Inequality,

\[
\|xf\|_{L^2(\mathbb{R})} \|\xi f\|_{L^2(\mathbb{R})} \geq \frac{\|f\|_{L^2(\mathbb{R})}^2}{4\pi}.
\]

It turns out that for \( 1 \leq p \leq 2 \) we can replace the \( L^2 \) norms to \( L^p \) norms on the left-hand side of Heisenberg’s Inequality.

**Proposition 4.1.** Under the assumptions of Theorem 2.1, we have

\[
\|xf\|_{L^p(\mathbb{R})} \|\xi f\|_{L^p(\mathbb{R})} \geq \frac{\|f\|_{L^2(\mathbb{R})}^2}{4\pi}
\]

for \( 1 < p \leq 2 \).

**Proof.** Repeat the proof of Heisenberg’s inequality, Theorem 2.1 with a modification at the step where we use Hölder’s inequality giving

\[
\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)|^2 \, dx = -2 \text{Re} \int_{\mathbb{R}} xf(x) \bar{f}'(x) \, dx \leq 2 \|xf\|_{L^p(\mathbb{R})} \|f'\|_{L^q(\mathbb{R})}
\]

for \( \frac{1}{p} + \frac{1}{q} = 1 \).

We now estimate the term \( \|f'\|_{L^q(\mathbb{R})} \). By the Hausdorff-Young Inequality, for \( 1 < p \leq 2 \) we obtain:

\[
\|f'\|_{L^q(\mathbb{R})} \leq \frac{p}{q} \frac{1}{p^*} \|(f')^\vee\|_{L^p(\mathbb{R})} \leq \|(f')^\vee\|_{L^p(\mathbb{R})}.
\]

By definition of \((f')^\vee\) and an integration by parts, we get

\[
\|f'\|_{L^q(\mathbb{R})} \leq \|(f')^\vee\|_{L^p(\mathbb{R})} = \left( \int_{\mathbb{R}} \int_{\mathbb{R}} f'(\xi)e^{2\pi i x \xi} \, d\xi \right)^{1/p} \int_{\mathbb{R}} \int_{\mathbb{R}} \left|2\pi i x f(\xi)e^{2\pi i x \xi} \right|^p \, d\xi \right)^{1/p} = \left( \int_{\mathbb{R}} \left|2\pi i (x) \right|^p \, dx \right)^{1/p} \|\xi f\|_{L^p(\mathbb{R})} = 2\pi \|\xi f\|_{L^p(\mathbb{R})}.
\]

which now gives the desired result of \( 4\pi \|xf\|_{L^p(\mathbb{R})} \|\xi f\|_{L^p(\mathbb{R})} \geq \|f\|_{L^2(\mathbb{R})}^2 \).

There are many more uncertainty inequalities in the \( L^p \) setting. Consider, for example, inequalities of the form

\[
\|xf\|_{L^p(\mathbb{R})} \|\xi f\|_{L^q(\mathbb{R})} \geq C \|f\|_{L^2(\mathbb{R})}, \quad a, b \in (0, \infty), \quad p, q \in [1, \infty], \quad \gamma \in (0, 1).
\]
Proposition 4.2. A necessary condition for (4.2) to hold is \( \gamma(a + \frac{1}{p} - \frac{1}{2}) = (1 - \gamma)(b + \frac{1}{q} - \frac{1}{2}) \).

The proof of this proposition is lengthy, and so we omit it here. The full details are found in [9].

5. Local Uncertainty Inequalities

Heisenberg’s inequality tells us that if \( f \) is highly localized, then \( \hat{f} \) cannot be concentrated near a single point. But it doesn’t say anything about \( \hat{f} \) being concentrated in a small neighborhood or perhaps a finite number of widely separated points.

As further motivation, let us consider an example in the context of signal analysis. Suppose \( \hat{f} \) being concentrated in a small neighborhood or perhaps a finite number of widely separated points.

Theorem 5.1 (Price, [9]). If \( 0 < \alpha < 1/2 \), there exists some constant \( K_\alpha \) such that for all \( f \in L^2(\mathbb{R}) \) and \( E \subseteq \mathbb{R} \) measurable,

\[
(\int_E |\hat{f}|^2)^{1/2} \leq K_\alpha |E|^{1/2} \| |x|\alpha f\|_{L^2(\mathbb{R})} \tag{5.1}
\]

Proof. For notational purposes, let \( \chi_r = \mathbb{1}_{|x|<r} \) and \( \chi_r' = 1 - \chi_r = \mathbb{1}_{|x|\geq r} \). Then

\[
\left( \int_E |\hat{f}|^2 \right)^{1/2} = \left\| \hat{f} \chi_r \right\|_{L^2(\mathbb{R})} \leq \left\| (f \chi_r)^\wedge \right\|_{L^1(\mathbb{R})} + \left\| (f \chi_r')^\wedge \right\|_{L^1(\mathbb{R})}
\]

where the inequality is due to a direct application of Minkowski’s inequality. There are two terms on the right-hand side of the above inequality. The first of the two terms can be bounded above by \( \left\| (f \chi_r)^\wedge \right\|_{L^\infty(\mathbb{R})} |E|^{1/2} \) where \( |E| \) denotes the Lebesgue measure of the set \( E \). We can obtain an upper bound for the second term by using Plancherel to get

\[
\left\| (f \chi_r')^\wedge \right\|_{L^2(\mathbb{R})} \leq \left\| (f \chi_r')^\wedge \right\|_{L^2(\mathbb{R})} = \| f \chi_r' \|_{L^2(\mathbb{R})}.
\]

Altogether this gives us

\[
\left( \int_E |\hat{f}|^2 \right)^{1/2} \leq \left\| \hat{f} \chi_r \right\|_{L^2(\mathbb{R})} \leq \left\| (f \chi_r)^\wedge \right\|_{L^\infty(\mathbb{R})} |E|^{1/2} + \| f \chi_r' \|_{L^2(\mathbb{R})} \tag{5.2}
\]

We can obtain even further upper bounds to (5.2) by observing

\[
\left\| (f \chi_r)^\wedge \right\|_{L^\infty(\mathbb{R})} \leq \| f \chi_r \|_{L^1(\mathbb{R})} \leq \| |x|^{-\alpha} \chi_r \|_{L^2(\mathbb{R})} \left\| |x|^\alpha f(x) \right\|_{L^2(\mathbb{R})} = \left( \int_{|x|<r} |x|^{-2\alpha} \chi_r \, dx \right)^{1/2} \left\| |x|^\alpha f(x) \right\|_{L^2(\mathbb{R})} \leq C_\alpha r^{1/2-\alpha} \left\| |x|^\alpha f(x) \right\|_{L^2(\mathbb{R})}
\]

and

\[
\| f \chi_r' \|_{L^2(\mathbb{R})} \leq \left\| |x|^{-\alpha} \chi_r' \right\|_{L^\infty(\mathbb{R})} \left\| |x|^\alpha f(x) \right\|_{L^2(\mathbb{R})} \leq r^{-\alpha} \left\| |x|^\alpha f(x) \right\|_{L^2(\mathbb{R})}
\]
which gives

\[
(5.3) \quad \left( \int_E |\hat{f}|^2 \right)^{1/2} \leq \left( C_\alpha |E|^{1/2} r^{1/2-\alpha} + r^{-\alpha} \right) \| |x|^\alpha f(x) \|_{L^2(\mathbb{R})}.
\]

Now choose \( r \) to minimize (5.3) (basic calculus will show that
\[
|\hat{f}(x)| \leq \left( C_\alpha |E|^{1/2} r^{1/2-\alpha} + r^{-\alpha} \right) \| |x|^\alpha f(x) \|_{L^2(\mathbb{R})}.
\]

There exists a proof for the case \( \alpha > 1/2 \). We omit the proof but the proof methods are similar to the proof just given:

**Theorem 5.2** (Price, [10]). If \( \alpha > 1/2 \), there exists some constant \( K_\alpha \) such that for all \( f \in L^2(\mathbb{R}) \) and \( E \subseteq \mathbb{R} \) measurable,

\[
\int_E |\hat{f}|^2 \leq K_\alpha |E| \| |x|^\alpha f \|_{L^2(\mathbb{R})} \left( \| |x|^\alpha f \|_{L^2(\mathbb{R})} \right).
\]

**Remark 5.3.** The local uncertainty principals give rise to global uncertainty inequalities as well. For example, Theorem 5.1 gives a global uncertainty inequality because

\[
\| f \|_{L^2(\mathbb{R})}^2 = \| \hat{f} \|_{L^2(\mathbb{R})}^2 \leq K'_\alpha |E| \| |x|^\alpha f \|_{L^2(\mathbb{R})} \left( \| |x|^\alpha f \|_{L^2(\mathbb{R})} \right).
\]

By again choosing \( r \) to minimize the above expression we obtain

\[
\| f \|_{L^2(\mathbb{R})}^2 \leq K''_\alpha |E| \| |x|^\alpha f \|_{L^2(\mathbb{R})} \left( \| |x|^\alpha f \|_{L^2(\mathbb{R})} \right).
\]

It should be noted that this global uncertainty constant, \( K''_\alpha \) is not the optimal constant \( 4\pi \) as obtained in Theorem 2.1.

### 6. Other generalizations, variations, modifications

There are hundreds of references and results of uncertainty inequalities in harmonic analysis.

In [4], De Bruijn derived a Heisenberg-like inequality using Hermite functions

\[
h_k(x) = \frac{2^{1/4}}{\sqrt{k!}} \left( \frac{-1}{2\sqrt{\pi}} \right)^k e^{\pi x^2} \frac{d^k}{dx^k} e^{-2\pi x^2}.
\]

Some facts of the Hermite functions (see [5]) are:

(1) \( \{ h_k \}_{k=0}^\infty \) is an orthonormal basis for \( L^2(\mathbb{R}) \).

(2) \( \hat{h}_k = i^{-k} h_k \).

(3) \( 2\sqrt{\pi} x h_k(x) = \sqrt{k+1} h_k+1(x) + \sqrt{k} h_k-1(x) \).

Given a function \( f \in L^2(\mathbb{R}) \) use property (1) to write \( f = \sum_{k=0}^\infty \langle f, h_k \rangle h_k \). Then by using properties (2) and (3) we can similarly obtain the expansions of \( xf \) and \( \xi \hat{f} \) to obtain

\[
\| xf \|_{L^2(\mathbb{R})}^2 + \| \xi \hat{f} \|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \sum_{k=0}^\infty (2k+1) |\langle f, h_k \rangle|^2.
\]
By Parseval, we know that $\|f\|_{L^2(\mathbb{R})}^2 = \sum_{k=0}^{\infty} |\langle f, h_k \rangle|^2$ which gives

$$\text{(6.1)} \quad \|xf\|_{L^2(\mathbb{R})}^2 + \|\hat{\xi f}\|_{L^2(\mathbb{R})}^2 \geq \frac{\|f\|_{L^2(\mathbb{R})}^2}{2\pi}.$$

Moreover, one can show that (6.1) holds with equality if and only if $f = ch_0(x) = e^{-\pi x^2}$.

This can be modified even further. Suppose that $f \in L^2(\mathbb{R})$ is odd (i.e. $f(-x) = -f(x)$). Then $\langle f, h_k \rangle = 0$ for $k$ even (since $h_k$ is an even function for even $k$). Then

$$\|xf\|_{L^2(\mathbb{R})}^2 + \|\hat{\xi f}\|_{L^2(\mathbb{R})}^2 = \frac{1}{2\pi} \sum_{k=0}^{\infty} (2k+1) |\langle f, h_k \rangle|^2 \geq \frac{3}{2\pi} \frac{\|f\|_{L^2(\mathbb{R})}^2}{2\pi}.$$

There are also what’s known as “qualitative” uncertainty principles. These are uncertainty principles which do not quantize estimates on $f$ or $\hat{f}$, yet still prohibit $f$ and $\hat{f}$ being sharply localized.

**Theorem 6.1** (Benedicks, [3]). If $f \in L^1(\mathbb{R}^d)$ and $|\Sigma(f)||\Sigma(\hat{f})| < \infty$, then $f \equiv 0$ where $\Sigma(f) = \{x : f(x) \neq 0\}$, $\Sigma(\hat{f}) = \{\xi : \hat{f}(\xi) \neq 0\}$.

**References**


