div, grad, and curl as linear transformations

Let $X$ be an open\footnote{With lots of hassle, one could do all this if $X$ is not open, but it is not worth obscuring the main ideas to do so. By the way, I suggest you ignore the footnotes on your first reading. The main text is meant to give the essential ideas involved, sometimes telling a white lie, but this is all corrected in the footnotes.} subset of $\mathbb{R}^n$. Let $SF_X$ denote the vector space of real valued functions on $X$ (i.e., scalar fields) and let $VF_X$ denote the vector space of vector fields on $X$. Colley defines maps\footnote{This is not quite true. Since the maps involve derivatives, you need differentiability of the functions and vector fields. Here are some options on how to correct this:}

grad: $SF_X \to VF_X$ and div: $VF_X \to SF_X$. If $n = 3$ Colley also defines curl: $VF_X \to VF_X$. The first thing to note is that div, grad, and curl are all linear transformations, since for example $\text{grad}(f + g) = \text{grad}f + \text{grad}g$ and $\text{grad}(cf) = c\text{grad}f$. Recall from Colley that $\text{div} \text{curl} = 0$ and $\text{curl} \text{grad} = 0$. So we will look at a little linear algebra where the composition of linear transformations is 0.

So suppose $T: U \to V$ and $S: V \to W$ are linear transformations and $ST = 0$. It may help to think of $T$ and $S$ as matrices whose product is 0, although this will not apply to the infinite dimensional examples we are studying here. Let $NS(S) = \{ \alpha \in V \mid S\alpha = 0 \}$ be the null space of $S$ which is a subspace of $V$. Let $T(U) = \{ \alpha \in V \mid \alpha = T\beta \text{ for some } \beta \in U \}$ be the range of $T$, which is also a subspace of $V$. Since $ST = 0$ we know that $T(U) \subset NS(S)$. If $S$ and $T$ were matrices, $NS(S)$ would be the null space of $S$ and $T(U)$ would be the column space of $T$. So $T(U) \subset NS(S)$ is the same as saying that each column of $T$ is in the null space of $S$ which I hope you can convince yourself of. (Note $0 = \text{Col}_i(ST) = \text{SCol}_i(T)$.

If $X \subset Y$ is a subspace then there is a vector space $X/Y$ called the quotient space which I will not describe\footnote{Okay, so I will describe it. Consider the equivalence relation on $X$ where we say $x \sim x'$ if $x - x' \in Y$. The vectors in $X/Y$ are equivalence classes under this equivalence relation. If $x \in X$, let $[x]$ denote its equivalence class in $X/Y$. We define $[x] + [x'] = [x + x']$ and $c[x] = [cx]$.}. If $X$ and $Y$ are both subsets of some $\mathbb{R}^n$, you can think of $X/Y$ as all vectors in $X$ which are perpendicular to $Y$. It turns out that $\dim(X/Y) = k$ if and only if there are $k$ linearly independent vectors $\beta_1, \beta_2, \ldots, \beta_k$ in $X$ so that any vector $\alpha$ in $X$ can be written uniquely as $\alpha = c_1\beta_1 + c_2\beta_2 + \cdots + c_k\beta_k + \beta$ where $\beta$ is in $Y$. For example if $X = \mathbb{R}^3$ and $Y$ is the plane with equation $x + 2y + z = 0$, then $\dim(X/Y) = 1$ since if for example $\beta_1 = (1, 0, 0)$ then any vector $(a, b, c)$ in $\mathbb{R}^3$ can be written $(a, b, c) = (a + 2b + c)\beta_1 + (-b - c, b, c)$ and $(-b - c, b, c)$ is in $Y$. Moreover there is no other way to write $(a, b, c)$ as a multiple of $\beta_1$ plus a vector in $Y$.

Now let us go back to our vector fields. We have vector spaces $H^1(X) = NS(\text{curl})/\text{grad}(SF_X)$ and $H^2(X) = NS(\text{div})/\text{curl}(VF_X)$. A consequence of something called DeRham cohomology is that $\dim(H^1(X))$ is the number of ‘tunnels’ running through $X$ and $\dim(H^2(X))$ is the number of ‘holes’ in $X$. Note that $H^1(X) = 0$ means that $NS(\text{curl}) = \text{grad}(SF_X)$ which means that a vector field on $X$ is conservative if and only if its curl is 0. So if $X \subset \mathbb{R}^3$ has no tunnels then a vector field on $X$ is conservative if and only if its curl is 0.

We say that $X$ is star shaped if there is a point $\alpha \in X$ so that for any $\beta \in X$ the entire line segment from $\alpha$ to $\beta$ is in $X$. I will show that if $X$ is star shaped and $G$ is a vector field on $X$ so $\text{curl} G = 0$, then $G = \text{grad} h$ for some $h$. In fact here is a formula for $h$, $h(x) = \int_0^1 G(\alpha + t(x - \alpha)) \cdot (x - \alpha) \, dt = \text{the work integral for } G \text{ on the line segment from } \alpha \text{ to } x$. By the product rule and the ability to differentiate under the integral sign,

\[
\begin{align*}
\partial h/\partial x_i &= \int_0^1 \partial(G(\alpha + t(x - \alpha)) \cdot (x - \alpha))/\partial x_i \, dt \\
&= \int_0^1 \partial(G(\alpha + t(x - \alpha))/\partial x_i \cdot (x - \alpha) + G(\alpha + t(x - \alpha)) \cdot e_i \, dt \\
&= \int_0^1 t\partial G/\partial x_i(\alpha + t(x - \alpha)) \cdot (x - \alpha) + G_i(\alpha + t(x - \alpha)) \, dt
\end{align*}
\]
But since \( \text{curl} G = 0 \) we know \( \partial G_i / \partial x_j = \partial G_j / \partial x_i \) so \( \text{grad} G_i = \partial G / \partial x_i \). So combining all this we get

\[
\frac{\partial h}{\partial x_i} = \int_0^1 t \partial G / \partial x_i (\alpha + t(x - \alpha)) \cdot (x - \alpha) + G_i (\alpha + t(x - \alpha)) \, dt \\
= \int_0^1 t \text{grad} G_i \cdot (x - \alpha) + G_i (\alpha + t(x - \alpha)) \, dt \\
= \int_0^1 dt G_i (\alpha + t(x - \alpha)) / dt \, dt = t G_i (\alpha + t(x - \alpha)) \bigg|_0^1 = G_i (x)
\]

So in the end, \( G = \text{grad} h \).

Note that we did not use dimension 3 at all above, except to say \( \text{curl} G = 0 \) implies \( \partial G_i / \partial x_j = \partial G_j / \partial x_i \). So in fact it proves in all dimensions that if \( G \) is a vector field with star shaped domain, then \( G \) is conservative if and only if \( \partial G_i / \partial x_j = \partial G_j / \partial x_i \) for all \( i \) and \( j \).

Let us do an example where this does not hold. Suppose \( X \) is the region inside the sphere \( x^2 + y^2 + z^2 = 9 \) and outside the cylinder \( x^2 + y^2 = 1 \). So \( X \) looks like a cored apple and it has one tunnel running through it, the (empty) core of the apple. Consider the vector field \( F(x, y, z) = (-y\hat{i} + x\hat{j})/(x^2 + y^2) \). Then

\[
\text{curl} F(x, y, z) = (\partial(x^2 + y^2)^{-1}) / \partial x - \partial(-y(x^2 + y^2)^{-1}) / \partial y) \hat{k} \\
= ((x^2 + y^2)^{-1} - x(x^2 + y^2)^{-2}(2x) + (x^2 + y^2)^{-1} - y(x^2 + y^2)^{-2}(2y)) \hat{k} = 0
\]

On the other hand, if \( F = \text{grad} g \) then \( \partial g / \partial x = -y(x^2 + y^2)^{-1} \), \( \partial g / \partial y = x(x^2 + y^2)^{-1} \), and \( \partial g / \partial z = 0 \). The last equation means that \( g \) is a function of \( x \) and \( y \) alone and does not depend on \( z \). Integrating the first equation we see that \( g(x, y) = \tan^{-1}(y/x) + C(y) \) for some function \( C \). Plugging into the second equation we get \( C(y) \) is a constant. So in polar coordinates, \( g(x, y) = \theta \). The problem is that this is not continuous, as you go around \( X \), \( g \) goes from 0 to \( 2\pi \) and then must immediately jump back from \( 2\pi \) to 0. So \( F \) is not conservative, there is no differentiable function \( g \) so that \( F = \text{grad} g \).

On the other hand, I claim that if \( G(x, y, z) \) is any vector field on \( X \) with \( \text{curl} G = 0 \) then there is a \( c \) and a \( g(x, y, z) \) so that \( G = cF + \text{grad} g \), so \( H^1(X) \) has dimension 1. This may be easiest to see using the cylindrical coordinate version given on page 219 of the text. Note \( F = (-r \sin \theta i + r \cos \theta j) / r^2 = \hat{e}_\theta / r \). If \( G = G_r \hat{e}_r + G_\theta \hat{e}_\theta + G_z \hat{e}_z \) then \( \text{curl} G = 0 \) means \( \partial G_z / \partial \theta = \partial r G_\theta / \partial z \), \( \partial G_z / \partial r = \partial G_\theta / \partial z \), and \( \partial G_r / \partial \theta = \partial G_\theta / \partial r \). We want to find \( c \) and \( g(r, \theta, z) \) so that \( G = cF + \text{grad} g \) which means \( G_r = 0 \), \( G_\theta = 0 \), \( G_z = 0 \). Letting \( \bar{G} = r G_\theta \) our equations become \( \partial G_z / \partial \theta = \partial G / \partial z \), \( \partial G_z / \partial r = \partial G_r / \partial z \), \( \partial G_r / \partial \theta = \partial G / \partial r \), \( G_r = \partial g / \partial r \), \( G_\theta = \partial g / \partial \theta \), \( G_z = \partial g / \partial z \). In other words, the vector field \( (G_r, G_\theta, G_z) \) in the \( r, \theta, z \) space has 0 curl. It is defined on the region \( 0 \leq \theta \leq 2\pi \), \( 1 \leq r \leq 3 \), \( -\sqrt{9-r^2} \leq z \leq \sqrt{9-r^2} \) which is star shaped and hence we may find a function \( h(r, \theta, z) \) so that \( \partial h / \partial r = G_r \), \( \partial h / \partial \theta = \bar{G} \), and \( \partial h / \partial z = G_z \).

Consider \( f(r, z) = h(r, 2\pi, z) - h(r, 0, z) \). Then

\[
\partial f(r, z) / \partial r = \partial h(r, 2\pi, z) / \partial r - \partial h(0, 0, z) / \partial r = G_r(r, 2\pi, z) - G_r(r, 0, z) = 0
\]

Similarly, \( \partial f(r, z) / \partial z = 0 \). So \( f(r, z) \) does not change when either \( r \) or \( z \) changes, so \( f(r, z) \) is some constant \( d \). Let \( g(r, \theta, z) = h(r, \theta, z) - \frac{d}{2\pi} \theta \). Then \( g(r, 2\pi, z) = g(r, 0, z) \) for all \( r \) and \( z \) so \( g \) is a well defined continuous function on \( X \). Moreover \( \text{grad} g = G - \frac{d}{2\pi} F \) which is what we wanted to show.