Math 241 Exam 1 Sample 5 Solutions

1. Given the following data:

\[ P = (1, 2, 3) \]
\[ Q = (4, 10, 2) \]
\[ \vec{a} = 1\hat{i} + 2\hat{j} - 2\hat{k} \]
\[ \vec{b} = -3\hat{i} + 2\hat{j} + 1\hat{k} \]

(a) Find a vector perpendicular to both \( \vec{PQ} \) and \( \vec{a} \). [10 pts]

**Solution:** We have \( \vec{PQ} = 3\hat{i} + 8\hat{j} - 1\hat{k} \) so a vector perpendicular to both would be \( \vec{PQ} \times \vec{a} = -14\hat{i} + 5\hat{j} - 2\hat{k} \).

(b) Find the projection of \( \vec{b} \) onto \( \vec{a} \). [5 pts]

**Solution:** We have

\[ Pr_{\vec{a}}\vec{b} = \frac{\vec{a} \cdot \vec{b}}{\vec{a} \cdot \vec{a}} = \frac{-2}{9} (1\hat{i} + 2\hat{j} - 2\hat{k}) \]

(c) Find the unit vector in the direction of \( \vec{PQ} \). [5 pts]

**Solution:** The answer is

\[ \frac{\vec{PQ}}{||\vec{PQ}||} = \frac{3\hat{i} + 8\hat{j} - 1\hat{k}}{\sqrt{3^2 + 8^2 + (-1)^2}} = \frac{3\hat{i} + 8\hat{j} - 1\hat{k}}{\sqrt{74}} \]
2. (a) Find the distance between the point \((3, 2, 1)\) and the plane \(2x - 3y + 10z = 20\). Simplify. [12 pts]

**Solution:** The normal vector for the plane is \(\vec{N} = 2\hat{i} - 3\hat{j} + 10\hat{k}\) and a point on the plane is \(P = (10, 0, 0)\). We have \(Q = (3, 2, 1)\) off the plane. The distance is then

\[
\text{dist} = \frac{|\vec{PQ} \cdot \vec{N}|}{||\vec{N}||} = \frac{|(-7\hat{i} + 2\hat{j} + 1\hat{k}) \cdot (2\hat{i} - 3\hat{j} + 10\hat{k})|}{\sqrt{2^2 + (-3)^2 + 10^2}} = \frac{|-14 - 6 + 10|}{\sqrt{113}} = \frac{10}{\sqrt{113}}
\]

(b) Find the symmetric equation for the line through the points \((2, -1, 4)\) and \((0, 1, 4)\). [8 pts]

**Solution:** The line (one version) has \(x = 2 - 2t, y = -1 + 2t\) and \(z = 4\). If we solve for \(t\) in the first two and set equal we get

\[
\frac{x - 2}{-2} = \frac{y + 1}{2}, \quad z = 4
\]
3. (a) Find the point where the line through \((0, 2, 1)\) and \((3, 4, 5)\) passes through the plane \(z = 0\). \([6 \text{ pts}]\)

**Solution:** The line has parametric equation \(x = 3t\), \(y = 2 + 2t\) and \(z = 1 + 4t\). This hits the plane when \(1 + 4t = 0\) or \(t = -\frac{1}{4}\) hence at \(x = 3(-1/4)\), \(y = 2 + 2(-1/4)\) and \(z = 0\).

(b) Sketch the VVF \(\vec{r}(t) = 2t \hat{i} + (2 - 4t) \hat{j} + t \hat{k}\) for \(0 \leq t \leq 2\). Indicate direction. \([6 \text{ pts}]\)

**Solution:** This is a straight line from \(\vec{r}(0) = 0 \hat{i} + 2 \hat{j} + 0 \hat{k}\) to \(\vec{r}(2) = 4 \hat{i} - 6 \hat{j} + 2 \hat{k}\). More or less like this:

(c) Give a parametrization of the oriented semi-ellipse shown here. \([8 \text{ pts}]\)

**Solution:** We have \(\vec{r}(t) = -2 \cos t \hat{i} + 4 \sin t \hat{j}\) with \(0 \leq t \leq \pi\).
4. (a) Assuming $a$ and $b$ are positive constants calculate the curvature of the ellipse 
$$\vec{r}(t) = a \cos t \hat{i} + b \sin t \hat{j}$$ at $t = \pi/2$.

**Solution:**
We have $\vec{v} = -a \sin t \hat{i} + b \cos t \hat{j}$ and $\vec{a} = -a \cos t - b \sin t$. Then $\vec{v} \times \vec{a} = ab \hat{k}$ and so the curvature is 
$$\kappa = \frac{||\vec{v} \times \vec{a}||}{||\vec{v}||^3} = \frac{ab}{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}$$ hence at $t = \pi/2$ we have $\kappa = \frac{ab}{a^3} = \frac{b}{a^2}$.

(b) Calculate the length of the curve $\vec{r}(t) = 2t \hat{i} + t^2 \hat{j} + \ln t \hat{k}$ for $1 \leq t \leq 2$. Simplify.

**Solution:** We have $\vec{r}'(t) = 2 \hat{i} + 2t \hat{j} + \frac{1}{t} \hat{k}$ and so

$$\text{Length} = \int_{1}^{2} ||\vec{r}'(t)|| \, dt$$
$$= \int_{1}^{2} \sqrt{4 + 4t^2 + \frac{1}{t^2}} \, dt$$
$$= \int_{1}^{2} \sqrt{(2t + \frac{1}{t})^2} \, dt$$
$$= \int_{1}^{2} 2t + \frac{1}{t} \, dt$$
$$= t^2 + \ln |t|^2 \bigg|_{1}^{2}$$
$$= (4 + \ln 4) - (1 + \ln 1)$$
$$= 3 + \ln 4$$
5. (a) Find the position vector satisfying $\ddot{a}(t) = 2 \hat{i} + 2 \hat{j}$, $\ddot{v}(0) = 1 \hat{i} - 2 \hat{j}$ and $\ddot{r}(1) = 3 \hat{i} + 5 \hat{j}$.

Solution: First:

$$\ddot{a}(t) = 2 \hat{i} + 2 \hat{j}$$

$$\ddot{v}(t) = \int 2 \hat{i} + 2 \hat{j} \, dt$$

$$\ddot{v}(t) = 2t \hat{i} + 2t \hat{j} + \dddot{C}$$

Then $\dddot{v}(0) = 0 \hat{i} + 0 \hat{j} + \dddot{C} = 1 \hat{i} - 2 \hat{j}$ so $\dddot{C} = 1 \hat{i} - 2 \hat{j}$ and so $\dddot{v}(t) = (2t + 1) \hat{i} + (2t - 2) \hat{j}$. Then

$$\ddot{r}(t) = (2t + 1) \hat{i} + (2t - 2) \hat{j}$$

$$\ddot{r}(t) = \int (2t + 1) \hat{i} + (2t - 2) \hat{j} \, dt$$

$$\ddot{r}(t) = (t^2 + t) \hat{i} + (t^2 - 2t) \hat{j} + \dddot{D}$$

Then $\dddot{r}(1) = 2 \hat{i} - 1 \hat{j} + \dddot{D} = 3 \hat{i} + 5 \hat{j}$ so $\dddot{D} = 1 \hat{i} + 6 \hat{j}$ and so $\dddot{r}(t) = (t^2 + t + 1) \hat{i} + (t^2 - 2t + 6) \hat{k}$.

(b) Sketch the plane $x + 2y + 3z = 12$. Label the three intercepts with their coordinates.

Solution: We have:

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