Math 241 Exam 2 Sample 3 Solutions

1. (a) We have \( \bar{u} = \frac{1}{\sqrt{2}} \hat{i} - \frac{1}{\sqrt{2}} \hat{j} \) and so

\[
D_{\bar{u}}h = \frac{1}{\sqrt{2}}(6xy) - \frac{1}{\sqrt{2}}3x^2
\]

\[
D_{\bar{u}}h(1,1) = \frac{1}{\sqrt{2}}(6) - \frac{1}{\sqrt{2}}3
\]

(b) We rewrite the plane as a level surface for a function of three variables and then take the gradient:

\[
f(x, y) = x^2 + y^2
\]

\[
z = x^2 + y^2
\]

\[
0 = x^2 + y^2 - z
\]

\[
g(x, y, z) = x^2 + y^2 - z
\]

\[
\nabla g(x, y, z) = 2x \hat{i} + 2y \hat{j} - \hat{k}
\]

\[
\nabla g(2,1,5) = 4 \hat{i} + 2 \hat{j} - \hat{k}
\]

We use this vector as the normal vector and the point \((2, 1, 5)\) to give the equation of the plane

\[
4(x - 2) + 2(y - 1) - 1(z - 5) = 0
\]

The point \((1, 0, -1)\) is on this plane because

\[
4(1 - 2) + 2(0 - 1) - (-1 - 5) = 0
\]
2. (a) We know $A = \frac{1}{2}bh$. The chain rule tells us

$$\frac{dA}{dt} = \frac{\partial A}{\partial b} \frac{db}{dt} + \frac{\partial A}{\partial h} \frac{dh}{dt} = \frac{1}{2} h(2) + \frac{1}{2} b(3)$$

and so at $b = 10$ and $h = 20$ we have

$$\frac{dA}{dt} = \frac{1}{2} (20)(2) + \frac{1}{2} (10)(3)$$

(b) We find $D(x, y) = (2y - 4)(-2) - (2x)^2$ and then test the points:

$D(0, 0) = (-4)(-2) = +$ so then $f_{xx}(0, 0) = -4$ so $(0, 0)$ is a relative maximum.

$D(2, 2) = (0)(-2) - 16 = -$ so $(2, 2)$ is a saddle point.

$D(-2, 2) = (0)(-2) - 16 = -$ so $(-2, 2)$ is a saddle point.
3. (a) We have

\[ \frac{9}{-9} \]

(b) We have

\[ \frac{4}{-2} \]

(c) \[ x^2 + y^2 = 9 \]

(d) \[ z = 1 + x^2 + y^2 \]
4. First we find \( f_x(x, y) = 2(x - 1) = 0 \) when \( x = 1 \) and \( f_y(x, y) = 2y = 0 \) when \( y = 0 \) and the point \((1, 0)\) is in the region so then \( f(1, 0) = 0 \).

On the boundary:

- For the circular part \( y^2 = 4 - x^2 \) so 
  \[ f = (x - 1)^2 + (4 - x^2) = x^2 - 2x + 1 + 4 - x^2 = -2x + 5 \] for \( 0 \leq x \leq 2 \) which attains a maximum of 5 (when \( x = 0 \)) and a minimum of 1 (when \( x = 2 \)).

- On the left vertical part \( x = 0 \) so 
  \[ f = (0 - 1)^2 + y^2 = y^2 + 1 \] for \(-2 \leq y \leq 2 \) which attains a maximum of 5 (when \( y = \pm 2 \)) and a minimum of 1 (when \( y = 0 \)).

Thus the maximum is 5 and the minimum is 0.
5. The constraint is the level curve for \( g(x, y) = x^2 + y^2 \) and so we have the system:

\[
\begin{align*}
y + 2 &= \lambda(2x) \\
x &= \lambda(2y) \\
x^2 + y^2 &= 4
\end{align*}
\]

Label these (a), (b) and (c).

Then (b) tells us \( \lambda = \frac{x}{2y} \) or \( y = 0 \).
If \( y = 0 \) then (b) tells us \( x = 0 \) but (c) tells us \( x = \pm 2 \) which contradicts itself so \( y \neq 0 \).
If \( \lambda = \frac{x}{2y} \) then into (a) gives us \( y + 2 = \left( \frac{x}{2y} \right) (2x) \) so that \( x^2 = y(y + 2) \) which goes into (c) to give us

\[
\begin{align*}
y(y + 2) + y^2 &= 4 \\
2y^2 + 2y - 4 &= 0 \\
2(y + 2)(y - 1) &= 0
\end{align*}
\]

So that \( y = -2 \) or \( y = 1 \). If \( y = -2 \) then (c) tells us \( x = 0 \) giving us \((0, -2)\) and if \( y = 1 \) then (c) tells us \( x = \pm \sqrt{3} \) giving us \((\pm \sqrt{3}, 1)\).

Then:
\[
\begin{align*}
f(0, -2) &= 0 \\
f(\sqrt{3}, 1) &= 3\sqrt{3} \\
f(-\sqrt{3}, 1) &= -3\sqrt{3}
\end{align*}
\]

So the maximum is \(3\sqrt{3}\) and the minimum is \(-3\sqrt{3}\).