1. True or false: Determine if the following are true or false. If false provide a counterexample. [20 pts]
   You do not need to prove anything about the counterexamples, just provide them.
   (a) The product of two irrational numbers is irrational.
   (b) If \( f : [a, b] \to \mathbb{R} \) is integrable on \([a, b]\) then it is differentiable on \((a, b)\).
   (c) The Taylor Polynomial for a polynomial equals that polynomial.
   (d) All polynomials are differentiable everywhere.

2. Suppose that \( f : (a, b) \to \mathbb{R} \) is uniformly continuous. Show that \( f \) is bounded. [15 pts]

3. Prove using Mathematical Induction that \( \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \). [15 pts]

4. Using only the Archimedian Principle prove that \( \left\{ \frac{\sqrt{n}}{\sqrt{k}} + \frac{1}{n^2} + 2 \right\} \) converges. [15 pts]

5. Prove that \( S \subseteq \mathbb{R} \) is dense iff \( \forall x \in \mathbb{R} \) there is a sequence \( \{x_n\} \) in \( S \) which converges to \( x \). [15 pts]

6. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is such that \( f(2) = 3 \), \( f'(2) = 0 \) and \( f''(x) \geq 3 \) for all \( x \in [0, 4] \). Find a lower bound on \( f(4) \). [15 pts]

7. Define \( f : [0, 2] \to \mathbb{R} \) by
   \[
   f(x) = \begin{cases} 
   x^2 & \text{if } 0 \leq x \leq 1 \\
   x & \text{if } 1 < x \leq 2 
   \end{cases}
   \]
   Using the limit definition of the derivative and the sequence definition of the limit prove that \( f'(1) \) does not exist. [10 pts]

8. Suppose \( f : \mathbb{R} \to \mathbb{R} \) is differentiable at \( x_0 \in (a, b) \) and \( f'(x_0) > 0 \). Prove there is a neighborhood \( I \) of \( x_0 \) such that for \( x \in I \) we have \( x < x_0 \Rightarrow f(x) < f(x_0) \) and \( x > x_0 \Rightarrow f(x) > f(x_0) \). [15 pts]

9. Suppose \( f(4) = 3 \) and \( F(x) = \int_0^x (x-t) f(t^2) \, dt \). Find \( F'(2) \). [10 pts]

10. Show that the Taylor expansion of \( f(x) = \sin(4x) \) around any \( x_0 \) converges for all \( x \). [10 pts]

11. Define \( f_n : [3, \infty) \to \mathbb{R} \) by \( f_n(x) = \frac{1}{n+x} \). Find the function \( f : [3, \infty) \to \mathbb{R} \) to which \( \{f_n\} \) converges pointwise and then show this convergence is uniform. [15 pts]

12. Define \( f : [0, 1] \to \mathbb{R} \) by \( f(x) = \frac{1}{x+1} \). Using our proof of the Weierstrass Approximation Theorem find the minimum degree polynomial which approximates \( f(x) \) uniformly within \( \epsilon = 0.1 \) and write this polynomial in \( \Sigma \) form. [15 pts]

13. Use Taylor Polynomials to prove that \( \int_0^1 e^{x^2} \, dx \geq \frac{4}{3} \). [15 pts]

14. Show that \( \sum_{k=1}^{\infty} \frac{3}{k^6} \) converges using the Weierstrass Convergence Criterion. [15 pts]