1. Suppose $a, b \in \mathbb{R}^+$. Prove that the Taylor expansion of $f(x) = a^{bx}$ around any $x_0$ converges for all $x$. **

2. Let $x_0 \in \mathbb{R}^+$ and define $f : \mathbb{R} - \{0\} \rightarrow \mathbb{R}$ by $f(x) = \frac{1}{x}$.
   (a) Find $p_n(x)$ around $x_0$. Use the geometric sum formula
   \[ \sum_{k=0}^{n} z^k = \frac{1 - z^{k+1}}{1 - z} \]
   to simplify $p_n(x)$ as much as possible.
   (b) Show directly that for every $n \in \mathbb{N}$ we have
   \[ f(x) - p_n(x) = \left(1 - \frac{x}{x_0}\right)^{n+1} \]
   (c) Use part (b) to prove that $f(x)$ equals its Taylor Expansion iff $x \in (0, 2x_0)$. **

3. For each of the following write down and expression for the remainder using both the Lagrange remainder and the Cauchy remainder formulas. For the former clarify what value $c$ could be. Simplify as much as possible (don’t integrate the latter).
   (a) $f(x) = \sqrt{x}$, $x_0 = 4$, $R_4(4.5)$. *
   (b) $f(x) = \cos(x)$, $x_0 = \pi$, $R_7(3)$. *

4. Suppose $f : \mathbb{R} \rightarrow \mathbb{R}$ has derivatives of all order and all are continuous. Furthermore suppose \{a_n\} is a bounded sequence and $f$ has the property that $|f^{(n)}(x)| \leq a_n$ for all $x$. Use the Cauchy Remainder Formula to show that $f$ equals its Taylor Expansion. **

5. Use the result of problem 4 from section 8.5 of the book to show that the CRT implies the LRT if $f^{(n+1)} : I \rightarrow \mathbb{R}$ is assumed to be continuous. That is, prove that
   \[ R_n(x) = \frac{1}{n!} \int_{x_0}^{x} f^{(n+1)}(t)(x - t)^n \, dt \]
   implies
   \[ R_n(x) = \frac{f^{(n+1)}(c_x)}{(n+1)!} (x - x_0)^{n+1} \text{ for some } c_x \text{ between } x_0 \text{ and } x \]

6. Show that the Approximation Theorem does not hold if we replace $I$ by a bounded open interval $(a, b)$ by showing that if $f(x) = \frac{1}{b-x}$ for all $x$ then $f : (a, b) \rightarrow \mathbb{R}$ cannot be uniformly approximated by polynomials. **

7. Find the minimum degree of the polynomial constructed via the Approximation Theorem for each of the following functions defined on $[0, 1]$ with $\epsilon = 0.1$.
   (a) $f(x) = 2 \mid x - \frac{1}{2} \mid$ *
   (b) $f(x) = 2^x$ **