(1) Prove that

a: \[ \sum_{k=0}^{n} k \binom{n}{k} t^k (1-t)^{n-k} = nt; \]

b: \[ \sum_{k=1}^{n} k(k-1) \binom{n}{k} t^k (1-t)^{n-k} = n(n-1)t^2; \]

c: \[ \sum_{k=0}^{n} k^2 \binom{n}{k} t^k (1-t)^{n-k} = nt(1 + (n-1)t); \]

(2) Using the Gram-Schmidt orthogonalization process calculate the first 4 Legendre polynomials starting from \( \{1, x, x^2, \ldots\} \).

(3) Using MATLAB and the recurrence relationships, generate the first 20 Legendre polynomials and the First 20 Chebyshev polynomials. Plot first 10 of each series on the interval \([-1, 1]\). (Submit the printouts of your codes, polynomials, and the graphs.)

(4) Find the Least Squares approximations for the curve Witch of Agnesi

\[ y = \frac{1}{x^2 + 1} \]

on the interval \([-5, 5]\] using

a: \( \{1, x, x^2, \ldots\} \),

b: Legendre polynomials,

c: Chebyshev polynomials and Chebyshev nodes

using MATLAB. Experiment with different degrees and different number of nodes. Compare the results.

Hint: to rescale the Legendre and Chebyshev polynomial to the interval \([a, b]\) use the formula

\[ P_{[a,b]}(x) = P_{[-1,1]} \left( \frac{x - \frac{1}{2}(a+b)}{\frac{1}{2}(b-a)} \right). \]