1. To compute $\sqrt{2}$ we consider the following Eudoxos iterations: starting with $x_0 = y_0 = 1$ we set $x_{n+1} = x_n + y_n$ followed by $y_{n+1} = x_{n+1} + x_n$. Then $y_n / x_n \rightarrow \sqrt{2}$.

(a) Explain how does the Eudoxos method work.

(b) How many iterations are required for an error $|y_n / x_n - \sqrt{2}| \leq 10^{-6}$?

Hint. Write the iterative procedure in the matrix form

\[
\begin{pmatrix}
x_{n+1} \\
y_{n+1}
\end{pmatrix} = A
\begin{pmatrix}
x_n \\
y_n
\end{pmatrix}.
\]

Find eigenvalues $\{\lambda_1, \lambda_2\}$ and eigenvectors $\{v_1, v_2\}$ of $A$. Then write the initial data

\[
(x_0, y_0)^T = av_1 + bv_2.
\]

Show that $|y_n / x_n - \sqrt{2}| < Cq^n$. Find $q$. Estimate the number of iterations that are necessary to achieve the required accuracy.

2. (a) Consider $f(x) = \frac{1}{2} x^T A x + b^T x + c$ where $A$ is a symmetric and positive definite $n \times n$ matrix. Find its minimizer along the line $y + \alpha p$ where $x \in \mathbb{R}^n$ ans $p \in \mathbb{R}^n$ are fixed, and $\alpha$ is the parameter.

(b) Show that for any nonsingular matrix $B \|Bx\| \geq \|x\|/\|B^{-1}\|$.

(c) Consider a quasi-Newton method $x_{k+1} = x_k + \alpha_k p_k$ where $p_k = -B_k^{-1} \nabla f_k$, where $B_k$’s are kept to be symmetric and positive definite. Show that if $\|B_k\|\|B_k^{-1}\| \leq M$ for all $k$, then $\cos \theta_k \geq M^{-1}$.

3. Consider the steepest descent method with exact line searches applied to $f(x) = \frac{1}{2} x^T A x + b^T x + c$ where $A$ is a symmetric and positive definite $n \times n$ matrix. Show that if the initial point $x_0$ is such that $x_0 - x^*$ is parallel to an eigenvector of $A$ then the steepest descent method will find the solution $x^*$ in one step.

4. Let $A$ be a symmetric positive definite matrix.

(a) Show that the set of eigenvectors of $A$ is conjugate w.r.t. $A$.

(b) Starting from an arbitrary basis $\{b_1, \ldots, b_n\}$ show how one can adjust the Gram-Schmidt procedure to obtain another basic $\{g_1, \ldots, g_n\}$ that is conjugate w.r.t. $A$. 