1. *To be done in Matlab.* Compute the first 3 positive zeros of

\[ f_k(x) = kx \sin x - \cos x = 0 \]  

for \( k = 0, 2, 1, \) and \( 5. \) Start with plotting the graphs of the functions \( f_k(x). \) Then:

**a.** Experiment with the Newton method. First set \( k = 1 \) and the initial points \( x_0 = 1, 2.1, 2.2. \) See where the iterations will converge. Then for each of the values of \( k \) and for each of the first 3 roots pick a good initial point. Make tables of \( |x_n - x_{n-1}|. \)

**b.** Show that Eq. (1) is equivalent to the series of equations \( T_m(x) = x, \) \( m = 0, 1, 2, \ldots, \) where

\[ T_m(x) = \frac{\pi}{2} - \arctan(kx) + \pi m. \]  

Find out for which \( k \)'s and \( m \)'s \( T_m(x) \) is a contraction on the corresponding interval \( I_m = \left[ \pi m - \frac{\pi}{2}, \pi m + \frac{\pi}{2} \right]. \) Use the fixed point iteration, if possible, to find the roots.

2. Consider the equation \( x^m = 0 \) where \( m \) is an integer greater than 1. Then \( x^* = 0 \) is the only root, and it is degenerate. Show that the Newton’s method converges \( q \)-linearly and find the \( q \)-factor.

3. Let \( f \) be a real-valued function of one variable, \( f(x^*) = 0, f'(x^*) \neq 0, f''(x^*) = 0 \) and \( f'' \) is Lipschitz continuous. Prove that then the Newton iteration converges cubically (i.e. \( q \)-superlinearly with \( q \)-order 3).

4. Show that the iterations of the secant method

\[ x_{k+1} = x_k - \frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})} \]

converge \( q \)-superlinearly with \( q \)-order \( (\sqrt{5} + 1)/2 \) (the golden ratio).

5. A function \( G \) is said to be Hölder continuous with exponent \( \alpha \) in \( \Omega \) if

\[ \|G(x) - G(y)\| \leq K\|x - y\|^\alpha \]  

for all \( x, y \in \Omega. \)

Let \( F(x) : \mathbb{R}^n \to \mathbb{R}^n \) be a continuous function, \( F(x^*) = 0, F'(x^*) \) be nonsingular, and \( F'(x) \) be Hölder continuous with exponent \( \alpha > 0 \) in a convex open set \( \Omega \subset \mathbb{R}^n \) containing \( x^*. \) Show that then the Newton iterates converge with \( q \)-order \( 1 + \alpha \) is \( x_0 \) is sufficiently close to \( x^*. \)