Midterm Exam.

1. Suppose you want to find the best approximation of a continuous function \( f(x) \) on \([0, 1]\) with a polynomial of degree \( n \) in the \( L_2 \) (least squares) sense. Outline a way to do it that is appropriate for large \( n \).

2. **True/False.** Let \( f(x) = \int_0^x \sin \left( \frac{1}{\ln(b\cdot x)} \right) \, dt, \ x \in [0, 2\pi] \). For any \( \epsilon > 0 \) one can find a polynomial \( p(x) \) such that \( \sup_{x \in [0, 2\pi]} |f(x) - p(x)| < \epsilon \). Please explain your answer.

3. **True/False.** Let \( f(x) = \int_0^x \sin \left( \frac{1}{\ln(2\cdot x)} \right) \, dt, \ x \in [0, 2\pi] \). For any \( \epsilon > 0 \) one can a Chebyshev interpolant \( p(x) \) such that \( \sup_{x \in [0, 2\pi]} |f(x) - p(x)| < \epsilon \). Please explain your answer.

4. Suppose you are applying \texttt{ifft} and then \texttt{fft} to the function \( f(x) = \sin (50x) \) on the interval \([0, 2\pi]\) for \( N = 32 \). What will be the outcome? 
   *Hint:* find where the frequencies of \( f(x) \) will be aliased.

5. Suppose an unknown analytic function \( f(x) \) is represented by a set of values at equidistant \( N \) points on the interval \([a, b]\), where \( N \) is large. How would you find \( f \) at intermediate points? How will the error of your interpolation depend on \( N \)?

6. Suppose \( f(x) \) is represented by a set of noisy data measured at equidistant \( N \) points where \( N \) is large. Suppose you need to estimate \( f'(x) \). How would you do it?

7. Suppose you need to estimate the first derivative of \( f \) at \( x = 0 \) accurately while you are given \( f(0.00), f(0.05), f(0.10), f(0.15), f(0.20) \). Please suggest two ways to do it. The error of your estimates will be proportional to \((0.05)^p\). What are \( p \)'s for your estimates?

8. Suppose you need to integrate \( f(x) = -31.83x^3 + 157.87x^2 - 468.26x - 4.01 + \cos (x + \sin 0.25x) \) over the interval \([0, 8\pi]\) with high precision. How can you do it using composite rules? How fast will the error of your integration decay with the size of subinterval \( h \)?

9. Let \( C_{n,0}(x) \) be a power series for \( f(x) \) obtained from the Padé power series \( R_{n+1,0}(x) \) using the method of economization on the interval \([-\alpha, \alpha] \). Consider the error functions \( E_1(x) = |f(x) - R_{n,0}(x)| \), \( E_2(x) = |f(x) - R_{n+1,0}(x)| \), and \( E_3(x) = |f(x) - C_{n,0}(x)| \) on \([-\alpha, \alpha] \). Compare the behavior of these error functions on the interval \([-\alpha, \alpha] \).

10. Suppose you need to estimate the integral \( I(f) = \int_{-1}^1 \frac{f(x) \, dx}{\sqrt{1-x^2}} \). Find weights and nodes for the 32-point quadrature rule that maximizes the degree of exactness. What will be the degree of exactness of this rule? 
   *Hint: to determine weights, find Chebyshev interpolant for \( f(x) \). Then observe that \( I(f) = I(f \cdot 1) = I(f \cdot T_0) \).

These concepts/formulas might be useful:

- Modulus of continuity \( \omega(\delta) = \sup_{x_1, x_2 \in [a,b], \ |x_1-x_2| < \delta} |f(x_1) - f(x_2)| \).
- Euler-Maclaurin formula for \( f \in C^{2m+2}[x_0, x_n] \):
  \[
  \int_{x_0}^{x_n} f(x) \, dx = T_n(f) + \sum_{i=1}^{m} \frac{B_{2i}}{(2i)!} h^{2i} (f^{(2i-1)}(x_0) - f^{(2i-1)}(x_n)) - \frac{B_{2m+2}}{(2m+2)!} (x_n - x_0) h^{2m+2} f^{(2m+2)}(\zeta). 
  \]
- Simpson’s rule: \( S(f) = \frac{b-a}{6} \left( f(a) + f \left( \frac{a+b}{2} \right) + f(b) \right) \), \( E_S(f) = -\frac{15}{128} \left( \frac{b-a}{2} \right)^5 f^{(4)}(\zeta) \).
- Trapezoidal rule: \( T(f) = \frac{b-a}{2} \left( f(a) + f(b) \right) \), \( E_T(f) = -\frac{1}{12} (b-a)^3 f^{(2)}(\zeta) \).
- \( N \)-point Chebyshev interpolation: \( P_{N-1}(x) = \frac{\sin((N-1)\phi)}{\sin(\phi)} + \sum_{i=1}^{N-1} c_i T_i(x) \).
- Orthogonality relationship for Chebyshev polynomials: \( \langle T_k, T_j \rangle = 0 \), if \( k \neq j \); \( = \frac{\pi}{2} \) if \( k = j \neq 0 \); \( = \pi \) if \( k = j = 0 \).