1. Interpolation by Spline Functions

Spline functions yield smooth interpolation curves that are less likely to exhibit the large oscillations characteristic for high degree polynomials. Splines are used e.g., in

- applications in graphics,
- numerical methods (e.g., for BVP),
- signal processing.

The simplest splines are the cubic splines. We will restrict ourselves to their considerations.

1.1. Theoretical foundations. Let

\[ \Delta := \{ a = x_0 < x_1 < \ldots < x_n = b \} \]

be a partition of the interval \([a, b]\).

Definition 1. A cubic spline \( S_\Delta \) on \( \Delta \) is a real function \( S_\Delta : [a, b] \rightarrow \mathbb{R} \) with the properties:

1. \( S_\Delta \in C^2[a, b] \),
2. \( S_\Delta \) coincides on every subinterval \([x_j, x_{j+1}]\), \( j = 0, 1, \ldots, n - 1 \) with a polynomial of degree at most 3.

Thus a cubic spline consists of cubic polynomials glued in such a manner that their values together with the values of their first two derivatives coincide at the interior nodes \( x_j, j = 1, 2, \ldots, n - 1 \).

Suppose a function \( f \) is given at the nodes \( x_j, j = 0, 1, \ldots, n \), i.e.,

\[ f(x_j) = f_j, \quad j = 0, 1, \ldots, n - 1. \]

We denote the vector \( \{f_0, \ldots, f_n\} \) by \( F \). We will denote a spline function that interpolates \( f \) at these points by

\[ S_\Delta(F; \cdot). \]

Note that \( S_\Delta(F; \cdot) \) is not uniquely defined by the sequence \( F \). Indeed, we have \( 4n \) coefficients of the cubic polynomials and \( n + 1 + 3(n - 1) = 4n - 2 \) conditions to satisfy. Hence we are free to assign two additional conditions that would allow us to determine \( S_\Delta(F; \cdot) \) uniquely. The typical conditions are:

1. zero second derivatives at the ends: \( S_\Delta''(F; a) = S_\Delta''(F; b) = 0 \),
2. periodicity: \( S_\Delta'(F; a) = S_\Delta'(F; b), S_\Delta''(F; a) = S_\Delta''(F; b) \),
3. assigned first derivatives at the ends: \( S_\Delta'(F; a) = f_0', S_\Delta'(F; b) = f_n' \).

A prerequisite for the second condition is that \( f_0 = f_n \).

There is a theorem proven e.g., in [1] that the spline function is unique for each of these cases.
1.2. Setting up a system of equations for a cubic spline. Throughout this section we will stick with a fixed partition \( \Delta := \{ a = x_0 < x_1 < \ldots < x_n = b \} \) and fixed vector of \( f \)'s values \( F = \{ f_0, \ldots, f_n \} \). We will use the following notations for the intervals between the nodes and their lengths:

\[
I_j := [x_j, x_{j+1}], \quad j = 0, 1, \ldots, n - 1, \quad \text{and} \quad h_{j+1} := x_{j+1} - x_j.
\]

The second derivatives of the spline functions at the nodes are called moments and denoted

\[
M_j := S''_{\Delta}(F, x_j), \quad J = 0, 1, \ldots, n.
\]

We will show that the spline function \( S_{\Delta}(F; x) \) is completely characterized by its moments, and the moments are found by solving a system of linear equations.

The second derivative of a spline function coincides with a linear function on each subinterval, and these linear functions can be described in terms of the moments \( M_j \):

(1) \[
S''_{\Delta}(F; x) = M_j \frac{x_{j+1} - x}{h_{j+1}} + M_{j+1} \frac{x - x_j}{h_{j+1}} \quad x \in I_j \equiv [x_j, x_{j+1}].
\]

Integrating Eq. (1) we obtain:

(2) \[
S'_{\Delta}(F; x) = -M_j \frac{(x_{j+1} - x)^2}{2h_{j+1}} + M_{j+1} \frac{(x - x_j)^2}{2h_{j+1}} + A_j,
\]

(3) \[
S_{\Delta}(F; x) = M_j \frac{(x_{j+1} - x)^3}{6h_{j+1}} + M_{j+1} \frac{(x - x_j)^3}{6h_{j+1}} + A_j (x - x_j) + B_j,
\]

for \( x \in I_j, j = 0, 1, \ldots, n \), where \( A_j \) and \( B_j \) are constants of integration. From \( S_{\Delta}(F; x_j) = f_j \) and \( S_{\Delta}(F; x_{j+1}) = f_{j+1} \) we obtain the following equations for \( A_j \) and \( B_j \):

\[
M_j \frac{h_{j+1}^2}{6} + B_j = f_j,
\]

\[
M_{j+1} \frac{h_{j+1}^2}{6} + A_j h_{j+1} + B_j = f_{j+1}.
\]

Hence

(4) \[
B_j = f_j - M_j \frac{h_{j+1}^2}{6},
\]

(5) \[
A_j = \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{h_{j+1}}{6} (M_{j+1} - M_j).
\]

This yields the following representation of the spline function in terms of its moments:

(6) \[
S_{\Delta}(F; x) = \alpha_j + \beta_j (x - x_j) + \gamma_j (x - x_j)^2 + \delta_j (x - x_j)^3 \quad \text{for} \quad x \in I_j,
\]
where

(7) \[ \alpha_j = f_j, \]
(8) \[ \gamma_j = \frac{M_j}{2}, \]
(9) \[ \beta_j = S_\Delta'(F; x_j) = -M_j \frac{h_{j+1}}{2} + A_j = \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{h_{j+1}}{6} (M_{j+1} + 2M_j), \]
(10) \[ \delta_j = \frac{S_\Delta''(F, x_j^+)}{6} = \frac{M_{j+1} - M_j}{6h_{j+1}}. \]

Now we need to calculate the moments \( M_j \). The continuity of \( S_\Delta'(F; x) \) at the interior nodes yields \( n-1 \) equations for the moments \( M_j \). Using Eqs. (3) and (5) we obtain

\[ S_\Delta'(F; x) = -M_j \frac{(x_{j+1} - x)^2}{2h_{j+1}} + M_{j+1} \frac{(x - x_j)^2}{2h_{j+1}} + \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{h_{j+1}}{6} (M_{j+1} - M_j). \]

Therefore, for \( j = 1, 2, \ldots, n-1 \) we have

\[ S_\Delta'(F; x_j^-) = \frac{f_j - f_{j-1}}{h_j} + \frac{h_j}{3} M_j + \frac{h_j}{6} M_{j-1}, \]
\[ S_\Delta'(F; x_j^+) = \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{h_{j+1}}{3} M_j - \frac{h_{j+1}}{6} M_{j+1}. \]

Since \( S_\Delta'(F; x_j^+) = S_\Delta'(F; x_j^-) \), we have

\[ \frac{h_j}{6} M_{j-1} + \frac{h_j + h_{j+1}}{3} M_j + \frac{h_{j+1}}{6} M_{j+1} = \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{f_j - f_{j-1}}{h_j} \]

for \( j = 1, 2, \ldots, n-1 \). These are \( n-1 \) equations for \( n+1 \) unknown moments. The remaining two equations can be gained from the boundary conditions. In the first case, \( S_\Delta''(F; a) = S_\Delta''(F; b) = 0 \) we set \( M_0 = M_n = 0 \). In the periodic case we set \( M_0 = M_n \) and

\[ \frac{h_n}{6} M_{n-1} + \frac{h_n + h_1}{3} M_n + \frac{h_1}{6} M_1 = \frac{f_1 - f_n}{h_1} - \frac{f_n - f_{n-1}}{h_n}. \]

**Exercise** Obtain additional two conditions for the case of assigned first derivatives at the end points:

\[ \frac{h_1}{3} M_0 + \frac{h_1}{6} M_1 = \frac{f_1 - f_0}{h_1} - f'_0, \]
\[ \frac{h_n}{6} M_{n-1} + \frac{h_n}{3} M_n = f'_n - \frac{f_n - f_{n-1}}{h_n}. \]

Eqs. (11), (12) and (13) can be written in the common format

\[ \mu_j M_{j-1} + 2M_j + \lambda_j M_{j+1} = d_j, \quad j = 1, 2, \ldots, n-1, \]
where

\[ \lambda_j := \frac{h_{j+1}}{h_j + h_{j+1}}, \]

\[ \mu_j := 1 - \lambda_j = \frac{h_j}{h_j + h_{j+1}}, \]

\[ d_j := \frac{6}{h_j + h_{j+1}} \left\{ \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{f_j - f_{j-1}}{h_j} \right\}. \]

In the case \( M_0 = M_n = 0 \) we set

\[ \lambda_0 = 0, \quad d_0 = 0, \quad \mu_n = 0, \quad d_n = 0. \]

In result, we arrive at the following system of equations

\[
\begin{bmatrix}
2 & \lambda_0 & 0 \\
\mu_1 & 2 & \lambda_1 \\
\mu_2 & \cdot & \cdot \\
\cdot & \cdot & \cdot \\
0 & \mu_n & 2
\end{bmatrix}
\begin{bmatrix}
M_0 \\
M_1 \\
\cdot \\
\cdot \\
M_n
\end{bmatrix}
= 
\begin{bmatrix}
d_0 \\
d_1 \\
\cdot \\
\cdot \\
d_n
\end{bmatrix}
\]

**Exercise** Write out the system of equations for the momenta for the periodic case and for the case of assigned first derivatives at the endpoints.

**Theorem 1.** The matrix in Eq. (17) is nonsingular for any partition \( \Delta \) of \([a, b]\).

**Proof.** We will denote the Matrix in Eq. (17) by \( A \). If follows from the definitions of \( \lambda_j \) and \( \mu_j \) that \( \mu_j + \lambda_j = 1, \mu_j \geq 0, \lambda_j \geq 0 \). Therefore, the matrix \( A \) is strictly diagonal dominant. Therefore, for any vector \( z \)

\[ \max_j |z_j| \leq \max_j |Az_j|. \]

(Check this!) Hence \( Az = 0 \) if and only if \( z = 0 \).
1.3. **A fast solver for tridiagonal matrices.** The matrix in Eq. (17) is tridiagonal. There is a fast solver for such kind of systems that involves $O(n)$ flops.

\[
\begin{align*}
q_0 &= -\frac{\lambda_0}{2}; \\
u_0 &= \frac{d_0}{2}; \\
\lambda_n &= 0; \\
\text{for } k = 1 : n & \\
p_k &= \mu_k q_{k-1} + 2; \\
q_k &= -\frac{\lambda_k}{p_k}; \\
u_k &= \frac{d_k - \mu_k u_{k-1}}{p_k}; \\
\end{align*}
\]

end for

\[
M_n = u_n;
\]

for $k = n - 1 : 0$

\[
M_k = q_k M_{k+1} + u_k;
\]

end for

1.4. **Convergence properties of cubic spline functions.**

**Definition 2.** The fineness of the given partition is

\[
\|\Delta\| := \max_j h_j.
\]

As we know, the interpolating polynomials do not necessarily converge to $f$ no matter how smooth $f$ is as the fineness of the partition tends to zero. The spline interpolants do converge to $f$ nicely under mild conditions.

First we will introduce some notations. We will denote by $F''$ the vector of the second derivatives of $f$ at the nodes. The vector of moments $M$ satisfies Eq. (17). We will write

\[
AM = d.
\]

The residual $r$ is defined as

\[
r := d - AF = A(M - F'').
\]

**Theorem 2.** Suppose the first derivatives of $f$ are assigned. If $f \in C^4[a, b]$ and $|f^{(4)}(x)| \leq L$ for $x \in [a, b]$, then

\[
\|M - F''\| \leq \|r\| \leq \frac{3}{4}L\|\Delta\|^2.
\]
Proof. By definition of the residual we have
\[ r_j = d_j - \mu_j f''(x_{j-1}) - 2f''(x_j) - \lambda_j f''(x_{j+1}) \]
\[ = \frac{6}{h_j + h_{j+1}} \left\{ \frac{f_{j+1} - f_j}{h_{j+1}} - \frac{f_j - f_{j-1}}{h_j} \right\} \]
\[ - \frac{h_j}{h_j + h_{j+1}} f''(x_{j-1}) - 2f''(x_j) - \frac{h_{j+1}}{h_j + h_{j+1}} f''(x_{j+1}). \]

Using Taylor’s expansion around \( x_j \) we obtain
\[ r_j = \frac{6}{h_j + h_{j+1}} \left( f' + \frac{h_j + 1}{2} f'' + \frac{h_j^2}{6} f''' + \frac{h_j^3}{24} f''''(\tau_1) \right) \]
\[ - f' + \frac{h_j}{2} f'' - \frac{h_j^2}{6} f''' - \frac{h_j^3}{24} f''''(\tau_2)) \]
\[ - \frac{h_j}{h_j + h_{j+1}} \left[ f'' - h_j f''' + \frac{h_j^2}{2} f''''(\tau_3) \right] - 2f'' - \]
\[ - \frac{h_{j+1}}{h_j + h_{j+1}} \left[ f'' + h_{j+1} f''' + \frac{h_{j+1}^2}{2} f''''(\tau_4) \right] \]
\[ = \frac{1}{h_j + h_{j+1}} \left[ \frac{h_j^3}{4} f''''(\tau_1) + \frac{h_{j+1}^3}{4} f''''(\tau_2) - \frac{h_j^3}{2} f''''(\tau_3) - \frac{h_{j+1}^3}{2} f''''(\tau_4) \right]. \]

Here all omitted arguments are \( x_j \) and \( \tau_j \in [x_{j-1}, x_{j+1}] \). Therefore, for \( j = 1, 2, \ldots, n-1 \)
\[ |r_j| \leq \frac{3}{4} L \frac{h_j^3 + h_{j+1}^3}{h_j h_{j+1}} = \frac{3}{4} L(h_j^2 - h_j h_{j+1} + h_{j+1}^2) \leq \frac{3}{4} L\|\Delta\|^2. \]

The first and the last intervals are handled in a similar manner. For them we have
\[ |r_0| \leq \frac{3}{4} L\|\Delta\|^2 \quad \text{and} \quad |r_n| \leq \frac{3}{4} L\|\Delta\|^2. \]

Since \( r = A(M - F'') \) and \( \|A\| \leq 1 \) we get
\[ \|M - F''\| \leq |r| \leq \frac{3}{4} L\|\Delta\|^2. \]

\[ \square \]

**Theorem 3.** Suppose \( f \in C^4[a, b] \) and \( |f^{(4)}(x)| \leq L \) for \( x \in [a, b] \). Let \( \Delta \) be a partition of the interval \([a, b]\) and \( K \) be a constant such that
\[ \frac{\|\Delta\|}{h_{j+1}} \leq K, \quad j = 0, 1, \ldots, n - 1. \]

If \( S_\Delta \) is the spline function which interpolates the values of \( f \) at the nodes of the partition \( \Delta \) and satisfies
\[ S_\Delta'(a) = f'(a) \quad \text{and} \quad S_\Delta'(b) = f'(b), \]
then there exist constants $c_k \leq 2$ independent of the partition $\Delta$, such that for $x \in [a, b]$

\begin{align*}
|f(x) - S_\Delta(x)| &\leq c_0 L \|\Delta\|, \\
|f'(x) - S'_\Delta(x)| &\leq c_1 L \|\Delta\|^2, \\
|f''(x) - S''_\Delta(x)| &\leq c_2 L \|\Delta\|^2, \\
|f'''(x) - S'''_\Delta(x)| &\leq c_3 L K \|\Delta\|.
\end{align*}

**Proof.** First we prove Eq. (21). For $x \in [x_{j-1}, x_j]$,

\[
S'''_\Delta(x) - f'''(x) = \frac{M_j - M_{j-1}}{h_j} - f'''(x)
\]

\[
= \frac{M_j - f'''(x_j)}{h_j} - \frac{M_{j-1} - f'''(x_{j-1})}{h_j} + \frac{f''(x_j) - f''(x) - [f'''(x_{j-1}) - f'''(x)]}{h_j} - f'''(x).
\]

Using the previous theorem and the Taylor expansion at $x$ we get

\[
|S'''_\Delta(x) - f'''(x)| \leq \frac{3}{2} L \|\Delta\| + \frac{1}{h_j} (x_j - x) f''' + \frac{(x_j - x)^2}{2} f'''(\eta_1)
\]

\[
- (x_{j-1} - x) f''' - \frac{(x_{j-1} - x)}{2} f'''(\eta_2) - h_j f'''
\]

\[
\leq \frac{3}{2} L \|\Delta\| + \frac{L \|\Delta\|^2}{2 h_j} = 2 L \|\Delta\| \cdot \frac{L \|\Delta\|^2}{h_j}
\]

\[
\leq 2 L K \|\Delta\|.
\]

Here $\eta_1, \eta_2 \in [x_{j-1}, x_j]$.

To prove Eq. (20) we observe that for every $x \in (a, b)$ there is a closest node $x_j = x_j(x)$. Assume without loss of generality that $x \leq x_j(x)$ so that $x \in [x_{j-1}, x_j]$ and $|x_j(x) - x| \leq h_j/2 \leq \|\Delta\|/2$. Then

\[
|f''(x) - S''_\Delta(x)| = |f''(x_j) - S''_\Delta(x_j) + \int_{x_j(x)}^{x_j} (f'''(t) - S'''_\Delta(t)) dt| 
\]

\[
\leq \frac{3}{4} L \|\Delta\|^2 + \frac{h_j}{2} L \|\Delta\|^2 \leq \frac{7}{4} L \|\Delta\|^2, \quad x \in (a, b).
\]

Next we prove Eq. (19). In addition to the boundary points $\xi_0 := a$ and $\xi_{n+1} := b$ there exist by Rolle’s theorem $n$ points $\xi_j \in (x_{j-1}, x_j)$ such that

\[
f'({\xi_j}) = S'_{\Delta}({\xi_j}).
\]

for any $x \in (a, b)$ there exists a closest one $\xi_j = \xi_j(x)$, and

\[
|x - \xi_j(x)| < \|\Delta\|.
\]
Therefore,

\[ |f'(x) - S\Delta(x)| = \left| \int_{\xi(x)}^x (f''(t) - S''_\Delta(t)) dt \right| \leq \|\Delta\| \frac{7}{4} L \|\Delta\|^2 = \frac{7}{4} L \|\Delta\|^3. \]

Finally we prove Eq. (18). We have

\[ |f(x) - S\Delta(x)| = \left| \int_{\xi_j(x)}^x (f'(t) - S'\Delta(t)) dt \right| \leq \|\Delta\| \frac{7}{4} L \|\Delta\|^2 = \frac{7}{8} L \|\Delta\|^4, \quad x \in [a, b]. \]

1.5. **Spline interpolation in Matlab.** The Matlab command `interp1` is capable of finding the cubic spline. The default mode is the linear interpolation, while adding 'spline' in the argument results in finding the cubic spline. Figure 1 is generated by the following sequence of commands.

```matlab
>> t=linspace(0,5,1000);
>> p=linspace(0,5,20);
>> flin=interp1(p,sin(p.^2),t);
>> fspl=interp1(p,sin(p.^2),t,'spline');
>> fig=figure;
>> hold on;
>> grid;
>> plot(t,sin(t.^2),'LineWidth',1)
>> plot(t,flin,'LineWidth',1,'color','k')
>> plot(t,fspl,'LineWidth',1,'color','r')
>> plot(p,sin(p.^2),'.b','MarkerSize',20)
>> axis tight
>> set(gca,'DataAspectRatio',[1 1 1])
```

**REFERENCES**

Figure 1. Linear interpolation and cubic spline interpolation.