1. \( \nabla H = (\nabla_v (x, y)) \)

C is a regular value of \( H \) if and only if for all \( v \in H^{-1}(c) \), \( \langle x, v \rangle \notin \nabla^\perp(c) \)

\( \nabla H \) is nondegenerate if \( \nabla_v (x, y) \) is nondegenerate.

This implies that \( c \) is a regular value of \( H \) if and only if it is a regular value of \( \nabla H(x) \).

2. Consider a curve \( \gamma(t) : \mathbb{R} \to N = H^{-1}(c) \) with \( \gamma(0) = (x_0, y) \), \( \gamma'(0) = (x, y) \), then

\[
\frac{d}{dt} \left(\gamma(t) \right)^2 + V(x) = 0,
\]

evaluated at \( t = 0 \), gives us

\[
\langle x, y \rangle + \langle \nabla V(x), y \rangle = 0.
\]

3. \( S = \{ (x, y, z) : z = g(x, y) \} \)

\[ dz = \frac{\partial g}{\partial x} \, dx + \frac{\partial g}{\partial y} \, dy \]

This is a plane tangent to \( S \) at the point \( P_0 \).

Therefore \( T_{P_0} S \) has the expression

\[
\vec{z} = \left. \frac{\partial g}{\partial x} \right|_{P_0} (x - x_0) + \left. \frac{\partial g}{\partial y} \right|_{P_0} (y - y_0) + z_0.
\]

This is also the expression of the graph of \( \partial g(x, y) \).

3. The expressions \( \varphi_1, \varphi_2 \) can be written down explicitly

\[
\varphi_1(x, y) = \frac{x}{1 + y}, \quad \varphi_2(x, y) = \frac{1}{1 + y}.
\]

It is a direct computation that \( \varphi_1 \circ \varphi_2 \) and \( \varphi_2 \circ \varphi_1 \) are differentiable functions from \( \mathbb{R} \) to \( \mathbb{R} \).

\( \varphi_1^{-1}(\mathbb{R}) \cup \varphi_2^{-1}(\mathbb{R}) \) covers \( S \).

This shows that \( (\mathbb{R}, \varphi_1^{-1}) \) and \( (\mathbb{R}, \varphi_2^{-1}) \) define a differentiable structure on \( S \).
4. Consider \( S \) defined as \( S = \sin S \).
Then \( S \) has the expression
\[
\begin{cases}
F = 0, & G = 0
\end{cases}
\]
The derivative is \( (\frac{dF}{dG}) \). We know from problem 1(b)
\[
\frac{dF}{dpS}, \frac{dG}{dpS}
\]
Then \( TpS + TpS \) implies \( \frac{dF}{dpG} \) are not linear for any \( p \in S+S \). This implies \( (\frac{dF}{dG}) \) nondegenerate
Implicit function theorem shows \( S+S \) is a regular curve.

5. Consider \( V_0 = U \circ f_{a}(u) \), then \( U V_0 \) covers \( V \cdot V \) is open in \( M \), it must have \( \dim V = n \)
Moreover \( \{(V_0, f_{a}|V_0)\} \) defines a differential structure on \( V \)
This implies \( V \) is a \( n \)-dim manifold.

6. Suppose \( f \) is not a constant on \( M \), then there must be two points
\( \bar{p}, \bar{q} \in M \) s.t. \( f(\bar{p}) \neq f(\bar{q}) \)
Find a curve \( \alpha: [0,1] \to M \) s.t. \( \alpha(0) = p, \alpha(1) = q \)
Then \( f(\alpha(t)) \) is not a constant function on \([0,1]\)
\[
\frac{d}{dt} \left(f(\alpha(t))\right) = df \cdot \alpha'(t) = 0 \quad \text{since} \quad df_{\alpha(t)} = 0 \quad \text{by \( \alpha(t) \in V \)}
\]
This is a contradiction.
So \( f \) is constant on \( M \).