1. (a) Let $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{(0,0)\})$ be defined by
$$\omega(x,y) = -\frac{y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy$$
Prove that $\omega$ is a closed form.
(b) Let $\gamma : [0,2\pi] \to \mathbb{R}^2 \setminus \{(0,0)\}$ be the smooth curve $\gamma(t) = (\cos t, \sin t)$.
Compute $\int_\gamma \omega$.
(we recall (see homework 6) that for any smooth curve $\gamma : [a,b] \to M$, we have $\int_\gamma \omega = \int_{[a,b]} \gamma^* \omega$.)
(c) Explain why this implies that $\omega$ cannot be exact (use Homework 6, Problem 9 (c)).

2. Let $\omega = x_3 \, dx_1 \wedge dx_2 + x_1 \, dx_2 \wedge dx_3 \in \Omega^2(\mathbb{R}^3)$.
(a) Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by
$$f(u,v) = (u^2, uv, v^2).$$
Compute $f^* \omega$ (write your result in the form $f^* \omega = g(u,v) du \wedge dv$).
(b) Same question with $f : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $f(u,v) = (1, v \sin u, u \cos v)$.

3. Let $M = \{(x,y,z) \in \mathbb{R}^3 ; z = x^2 + y^2, \ z < 1\}$ be a smooth 2-manifold in $\mathbb{R}^3$ ($M$ is an open subset of the paraboloid).
(a) Show that $M$ is orientable.
(b) Let $\omega = xdy \wedge dz + ydz \wedge dx + zd\gamma \wedge dy \in \Omega^2(\mathbb{R}^3)$. Compute
$$\int_M \omega.$$ 

4. Let $S$ be a regular surface in $\mathbb{R}^3$. We want to show that $S$ is orientable if and only if there exists a smooth map $n : S \to \mathbb{R}^3$ such that for all $p \in S$, we have $||n(p)|| = 1$ and $n(p)$ orthogonal to $T_p S$ (we say that $n(p)$ is a unit normal vector to $S$ at $p$).
(a) First, assume that such a smooth map $n$ exists. Let $\omega_0 \in \Lambda^3(\mathbb{R}^3^*)$ be
defined by $\omega_0 = dx_1 \wedge dx_2 \wedge dx_3$. Show that the interior product
$$\omega(p) = i_{n(p)}\omega_0$$
defines a differential 2-form on $S$ such that $\omega(p) \neq 0$ for all $p \in S$ and that
this implies that $S$ is orientable.

(b) Assume now that $S$ is orientable. Let $p$ be a point on $S$, and assume that
$p = f_\alpha(u_0, v_0)$ for some parametrization $f_\alpha$. We define the cross-product
$$N_\alpha(p) = \frac{\partial f_\alpha}{\partial u}(u_0, v_0) \times \frac{\partial f_\alpha}{\partial v}(u_0, v_0) \in \mathbb{R}^3.$$

(i) Explain why $N_\alpha(p)$ is orthogonal to $T_pS$, and why $N_\alpha(p) \neq 0$.

(ii) We now set
$$n(p) = \frac{N_\alpha(p)}{||N_\alpha(p)||}.$$
Show that this defines a smooth map $n : f_\alpha(U_\alpha) \to \mathbb{R}^3$ with $n(p)$
normal unit vector (first show that $p \mapsto N_\alpha(p)$ is a smooth map).

In order to conclude, it remains to show that this construction does not
depend on the parametrization. Assume that $p \in f_\alpha(U_\alpha) \cap f_\beta(U_\beta)$ for
two parametrizations $f_\alpha$ and $f_\beta$ defining the same orientation (that is
det$(J(f_\beta^{-1} \circ f_\alpha)) > 0$). The construction above yields two vectors $N_\alpha(p)$
and $N_\beta(p)$ (depending on whether we use $f_\alpha$ or $f_\beta$ in the cross product).

(iii) Show that $N_\beta(p) = \lambda N_\alpha(p)$ for some $\lambda > 0$ (the positivity of $\lambda$ is very
important here) and that
$$n(p) = \frac{N_\alpha(p)}{||N_\alpha(p)||} = \frac{N_\beta(p)}{||N_\beta(p)||}.$$
(Hint: Use the fact, proved in class, that the matrix $P = J(f_\beta^{-1} \circ f_\alpha)$
is the change of coordinate matrix from the basis \{\frac{\partial f_\alpha}{\partial u}, \frac{\partial f_\alpha}{\partial v}\} to the
basis \{\frac{\partial f_\beta}{\partial u}, \frac{\partial f_\beta}{\partial v}\}).

5. Using the result of Problem 4, show that if $S$ is a regular surface of $\mathbb{R}^3$ defined
by $S = F^{-1}(c)$ where $c$ is a regular value of a smooth function $F : \mathbb{R}^3 \to \mathbb{R}$,
then $S$ is orientable.