1. Let $M$ be a compact 3-manifold in $\mathbb{R}^3$ with boundary $\partial M$. We recall that $\partial M$ is an orientable 2-manifold in $\mathbb{R}^3$, and so there exists an outward unit normal vector $n(p)$ defined on $\partial M$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R}^3 \to \mathbb{R}$ be two differentiable functions. Prove that (first Green’s identity)
\[
\int_M \text{grad } f \cdot \text{grad } g \omega + \int_M \Delta g \omega = \int_{\partial M} f (\text{grad } g \cdot n) dA
\]
where $\omega = dx_1 \wedge dx_2 \wedge dx_3$ is the usual volume element of $\mathbb{R}^3$ and $dA$ is the area form on $\partial M$. Here, $\Delta g$ denotes the laplacian of $g$ and is defined by $\Delta g = \text{div grad } g = \sum_{i=1}^3 \frac{\partial^2 g}{\partial x_i^2}$.

Hint: Define $H = f \text{grad } g : \mathbb{R}^3 \to \mathbb{R}^3$ and use the divergence theorem.

2. Let $U$ be a star-shaped open subset of $\mathbb{R}^3$ (this means that there exists $x^0 \in U$ such that for all $x \in U$, the segment $[x^0, x] = \{tx^0 + (1-t)p; t \in [0,1]\}$ is contained in $U$).

(a) Show that $U$ is contractible.

(b) Given $\omega = f(x)dx_1 \wedge dx_2 \wedge dx_3$ 3-form, define
\[
\eta(x) = \left( \int_0^1 t^2 f(tx + (1-t)x^0) \, dt \right) [(x_1-x_1^0)dx_2 \wedge dx_3 - (x_2-x_2^0)dx_1 \wedge dx_3 + (x_3-x_3^0)dx_1 \wedge dx_2].
\]
Show that $d\eta = \omega$.

3. Let $U$ be an open subset of $\mathbb{R}^n$ such that $M = \overline{U}$ is a differentiable manifold with boundary $\partial M \neq \emptyset$.

(a) Let $\omega = x_1dx_2 \wedge \cdots \wedge dx_n \in \Omega^{n-1}(M)$ and $\alpha = j^*\omega \in \Omega^{n-1}(\partial M)$ (where $j : \partial M \to M$ is the usual inclusion map $j(x) = x$). Show that
\[
\int_{\partial M} \alpha > 0.
\]

(note that $d\omega = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$).

(b) Let $f : M \to \partial M$ be a smooth function such that $f(x) = x$ for all $x \in \partial M$. Show that $d(f^*\alpha) = 0$ and $j^*(f^*\alpha) = \alpha$.

(c) Using (a) and (b), together with Stokes theorem, show that such a function $f$ cannot exists.
(d) Deduce the following result (Brouwer fixed point theorem):

**Theorem** Let $B$ be the ball $\{x \in \mathbb{R}^n ; ||x|| \leq 1\}$. If $g$ is a smooth function $g : B \to B$, then there exists $x_0 \in B$ such that $g(x_0) = x_0$ (such a point is called a fixed point of $g$).

**Hint:** Show that if $g$ does not have a fixed point in $B$, then there exists a function $f : B \to \partial B$ such that $f(x) = x$ for all $x \in \partial D$. Such a function can be constructed as follows: Let $f(x)$ be the intersection of the half line starting in $g(x)$ and passing through $x$ with $\partial B$. Find a formula for $f$ to check that it is smooth and use (c) to conclude.

4. (a) Let $M$ be a compact orientable $n$-manifold without boundary (i.e. $\partial M = \emptyset$) and let $\omega$ be a differential $(n-1)$-form on $M$. Show that there exists a point $p \in M$ such that $d\omega = 0$.

(b) Using (a), show that there exists no immersion $f : S^1 \to \mathbb{R}$ of the unit circle into the real line $\mathbb{R}$.

5. Let $F = (f_1, f_2, f_3)$ be a smooth vector field defined in an open subset $U$ of $\mathbb{R}^3$. Assume that $U$ is smoothly contractible to a point. Show that if $\text{div} \ F = 0$ in $U$, then there exists a vector field $G = (g_1, g_2, g_3)$ such that $F = \text{curl} \ G$. 