1. Show that there is at most one smooth solution to the equation

$$u_{tt} + cu_t - u_{xx} = f$$

in the domain $(0, 1) \times (0, \infty)$ with initial conditions $u(x, 0) = g(x)$, $u_t(x, 0) = h(x)$ and boundary conditions $u(0, t) = u(1, t) = 0$ (you can start by assuming $c \geq 0$, but the case $c < 0$ is required also).

2. Let $u$ be a $C^2$ solution of $u_{tt} - \Delta u = 0$ in $\mathbb{R}^2 \times (0, \infty)$, with

$$u(x, 0) = 0, \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R}^2$$

where $g$ is a smooth function satisfying $g(x) = 0$ for $|x| > a$.

(a) Show that there exists a constant $C$ such that $|u(x, t)| \leq \frac{C}{t}$ for $t \geq 2(|x| + a)$. **Hint:** Use Poisson formula

(b) Show that $\lim_{t \to \infty} tu(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} g(y) \, dy$ for all $x \in \mathbb{R}^2$.

3. Use characteristics to solve the linear equation $x_1 u_{x_1} + x_2 u_{x_2} = 2u$ in $\{(x_1, x_2), \; x_1 \in \mathbb{R}, \; x_2 > 1\}$ with boundary condition $u(x_1, 1) = g(x_1)$ (and sketch the projected characteristics $s \mapsto x(s)$).

4. Consider the equation

$$-(u_{x_1})^2 + (u_{x_2})^2 + x_2^2 = 0, \quad x_1 \in \mathbb{R}, \; x_2 > 0$$

with boundary condition $u(x_1, 0) = g(x_1)$ where $g$ is a $C^1$ function satisfying $g'(y) > 0$ for all $y$.

(a) Find explicitly the characteristics $x_1(s), \; x_2(s)$ starting from the point $(y, 0)$ (assuming $u_{x_2}(x, 0) \geq 0$).

(b) In the case $g(x_1) = x_1$, sketch the characteristics and write down an explicit solution (your answer may depend on the function $I(s) = \int_0^s \sin^2(r) \, dr$).
5. Find a smooth solution of

\[ u_t + (u_x)^4 = 0 \quad x \in \mathbb{R}, \ t > 0 \]

with initial condition \( u(x, 0) = \frac{3}{4} x^{4/3} \).