1. Fix $C > 0$, and let $K_C$ denote the subset of $L^2(\mathbb{R}, m)$, $m$ denoting the Lebesgue measure, consisting of functions satisfying
\[
\int_{\mathbb{R}} |f(x)|dm(x) \leq C.
\]
(i.) Show that $K_C$ is closed in $L^2(\mathbb{R}, m)$.
(ii.) Using the parallelogram law, show that for any $f \in L^2(\mathbb{R}, m)$, there exists a unique $f_C \in K_C$ so that
\[
\|f - f_C\|_2 = \inf\{\|g - f\|_2 : g \in K_C\}
\]
and then show that $\lim_{C \to \infty} f_C = f$ in $L^2(\mathbb{R}, m)$.

2. (a) Show that if $f$ is Lebesgue integrable then the set \{\(x : f(x) \neq 0\}\} is a $\sigma$–finite set.
(b) Show that if $f$ is Lebesgue integrable and positive a.e. on a Lebesgue measurable set $E$, and if
\[
\int_E f dm = 0, \text{ then } m(E) = 0.
\]

3. Let $m$ be the Lebesgue measure on $(0, \infty)$. Suppose that $(x^p + \frac{1}{x^p})f \in L^2((0, \infty), dm)$, where $p > 1/2$. Show that $f \in L^1((0, \infty), dm)$.

4. (a) Prove that for any Lebesgue integrable functions $f$ and $g$,
\[
0 \leq |f(x) - g(x)| - |f(x)| + |g(x)| \leq 2|g(x)| \quad a.e.
\]
(b) Let $m$ denotes the Lebesgue measure on $\mathbb{R}$. Suppose that \{\(f_n\)\} is a sequence in $L^1(\mathbb{R}, dm)$ with $\|f_n\|_1 \leq 1$ for all $n$ and
\[
\lim_{n \to \infty} f_n(x) = f(x) \quad a.e.
\]
Prove that in this case
\[
\lim_{n \to \infty} (\|f - f_n\|_1 - \|f_n\|_1 + \|f\|_1) = 0.
\]
(c) Suppose, in addition to the assumptions made in (b), that $f_n(x) \geq 0$ a.e. for all $n$, and
\[
\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) dm = \int_{\mathbb{R}} f(x) dm
\]
Prove that in this case
\[
\lim_{n \to \infty} \|f_n - f\|_1 = 0
\]

5. Guess the limit as $n \to \infty$ of
\[
\int_0^n (1 + x/n)^n e^{-2x} dx.
\]
Prove your guess is correct.

6. For $f \in L^2([0, 1])$, define a measurable function $Vf$ by
\[
Vf(x) = \int_0^x f(t) dt.
\]
(a) Show that \( Vf \in L^2([0, 1]) \) and that \( \|Vf\|_2 \leq \frac{1}{2} \|f\|_2 \).

(b) Suppose \( f_n \in L^2([0, 1]) \) and \( \|f_n\|_2 \leq 1 \) for all \( n \). Prove that \( \{Vf_n\} \) has a convergent subsequence with respect to the \( L^2 \) norm.

7. Fix \( f \in L^1(\mathbb{R}, m) \) where \( m \) is the Lebesgue measure on \( \mathbb{R} \), and define \( F(x) = \int_{\mathbb{R}} f(t) \frac{\sin xt}{t} dt \).

(a) Prove that \( F \) is differentiable on \( \mathbb{R} \), and find \( F' \).

(b) Is \( F \) absolutely continuous on each compact subinterval of \( \mathbb{R} \)?

8. Let \( \{f_n\} \) be a sequence of Lebesgue measurable functions on \([0, 1]\). Suppose that

\[ 0 \leq f_n \leq 1, \quad \text{for all} \quad n \quad \text{and} \quad \lim_{n \to \infty} f_n = c \quad \text{in measure.} \]

Prove that

\[ \lim_{n \to \infty} \int_0^1 (1 - \frac{1}{n})^n f_n dm = e^{-c}. \]

9. Produce an explicit example of a continuous function of two variable \( x \geq 1, \ t \geq 1 \), such that

\[ \int_1^\infty \int_1^\infty f(x, t)dx dt \neq \int_1^\infty \int_1^\infty f(x, t)dt dx \]

although both integrals exist separately.

10. Solve the following problems from the textbook. 5.8, 5.11, .5.14, 5.48, 5.54.

11. Let \( m \) denote the Lebesgue measure on \([0, 1]\). Let \( \{f_n\}_{n \in \mathbb{N}} \) be a sequence of Lebesgue measurable functions defined on \([0, 1]\) such that for each \( n \geq 1 \), \( 0 \leq f_n \leq M \) for a positive constant \( M \). Moreover, assume that \( \|f_n\|_1 = \int_0^1 f_n(x) dm(x) = 1 \) for all \( n \geq 1 \). Consider a sequence \( \{a_n\} \) such that \( a_n \geq 0 \) for all \( n \geq 1 \), and assume that \( \sum_{n=1}^{\infty} a_n = \infty \).

a. State Egoroff’s theorem.

b. Prove that there exists a subset \( A \) of positive measure in \([0, 1]\) such that \( \sum_{n=1}^{\infty} a_n f_n(x) = \infty \) for each \( x \in A \).

12. Fix positive real numbers \( p, q, a \), satisfying \( 1 \leq p < q < \infty \) (there is no condition on \( a \) except \( a > 0 \)), and give the interval \([0, a]\) the usual Lebesgue measure (of total mass \( a \)). Let \( \|\|_p \) and \( \|\|_q \) denote the norms for \( L^p([0, a]) \) and for \( L^q([0, a]) \), respectively.

a. Show that there is a constant \( C_a > 0 \) such that if \( f \in L^q([0, a]) \), then \( f \in L^p([0, a]) \), and \( \|f\|_p \leq C_a \|f\|_q \). Find the minimal value of \( C_a \), and show that it cannot be improved.

b. Show that \( C_a \to \infty \) as \( a \to \infty \).

c. Regardless of the value of \( a > 0 \), show that there is NO constant \( C > 0 \) such that if

\[ f \in L^p([0, a]) \cap L^q([0, a]) \]

then \( \|f\|_q \leq C \|f\|_p \).
13. Assume that \( n \geq 1 \) is an integer, and let \( f \in \bigcap_{n=1}^{\infty} L^n([0, 1]) \). Prove that if \( \sum_{n=1}^{\infty} \|f\|_{L^n([0, 1])} < \infty \), then \( f = 0 \) a. e.

14. Let \( f \in L^1(\mathbb{R}) \).
   (a) Determine
   \[
   \lim_{x \to 0} \int_{\mathbb{R}} |f(t + x) + f(t)| \, dt.
   \]
   (b) Determine
   \[
   \lim_{x \to \infty} \int_{\mathbb{R}} |f(t + x) + f(t)| \, dt.
   \]

15. Let \( f \in AC[0, 1] \) and \( f > 0 \). Prove that \( 1/f \in AC[0, 1] \).

16. Let \( f \in L^p(\mathbb{R}) \), \( 1 \leq p < \infty \), \( \alpha > 0 \), and define
   \[
   E_{\alpha}(f) = \{ x \in \mathbb{R} : |f(x)| > \alpha \}.
   \]
   (i) Show that \( E_{\alpha} \) has finite Lebesgue measure.
   (ii) Use (i) to show that every \( f \in L^p(\mathbb{R}) \), \( 1 \leq p \leq 2 \), can be decomposed as \( f = f_1 + f_2 \) where \( f_1 \in L^1(\mathbb{R}) \) and \( f_2 \in L^2(\mathbb{R}) \).

17. Let \( \{f_n\} \) be a sequence of measurable functions which converges a.e. to \( f \) on \( \mathbb{R} \), and suppose there exists \( g \in L^2(\mathbb{R}) \) such that for all \( n \geq 1 \) \( |f_n| \leq g \) a.e. on \( \mathbb{R} \).
   Given \( \epsilon > 0 \), prove that there is a measurable subset \( A \subset \mathbb{R} \) such that \( m(A) < \epsilon \) and \( f_n \to f \) uniformly on \( A^c \).

18. Let \( \{A_n\}_{n \geq 1} \) be a sequence of Lebesgue measurable subsets of \([0, 1]\). Assume that 1 is a limit point of the sequence \( \{m(A_n)\}_{n \geq 1} \) where \( m \) denotes the Lebesgue measure on \([0, 1]\). Prove that there exists a subsequence \( A_{n_k} \) such that
   \[
   m(\cap_{k=1}^{\infty} A_{n_k}) > 0.
   \]

19. Let \( f, f' \in L^1(\mathbb{R}) \) and assume that \( f \) is absolutely continuous on each bounded interval \( I \) in \( \mathbb{R} \). Prove that
   \[
   \int_{-\infty}^{\infty} f'(x) \, dx = 0.
   \]

20. Given \( a < b \), denote by \( m \) the Lebesgue measure on \([a, b]\). Let \( f \) be a positive and Lebesgue measurable function defined on \([a, b]\) such that
   \[
   f \in L^r([a, b], dm) = \{ f : [a, b] \to (0, \infty) : \int_{a}^{b} f(x)^r \, dx < \infty \}
   \]
   for some \( r > 0 \).
   a. Show that \( f \in L^s([a, b], dm) \) for each \( 0 < s \leq r \).
   b. Show that \( \lim_{s \to 0^+} \int_{a}^{b} f(x)^s \, dx = b - a \) and conclude that
   \[
   \lim_{s \to 0^+} \left( \int_{a}^{b} f(x)^s \, dx \right)^{1/s} = \infty \text{ if } b - a > 1
   \]
and

\[ \lim_{s \to 0^+} \left( \int_a^b f(x)^s \, dx \right)^{1/s} = 0 \text{ if } b - a < 1. \]