1) The goal of this problem is to construct a non-measurable (in the sense of Lebesgue) subset \( P \subset (0, 1) \).

Consider the following relation for \( x, y \in (0, 1) \), \( x \sim y \) if and only if \( x - y \in \mathbb{Q} \).

a) Verify that \( \sim \) is an equivalence relation on \((0, 1)\).

b) For each equivalence class \( C \) of \( \sim \), choose a representative \( y \in (0, 1) \), and let \( P \) be the collection of all such \( y \). (Note that \( P \) is well defined by the Axiom of Choice (see page 76 of the textbook)). Let \( \{r_n\}_{n=1}^{\infty} = (-1, 1) \cap \mathbb{Q} \), and define \( P_n = r_n + P \) for each \( n \geq 1 \).

b.1 Prove that \( P_n \cap P_m = \emptyset \) if \( m \neq n \).

b.2 Prove that \((0, 1) \subset \bigcup_{n=1}^{\infty} P_n \subset (-1, 2)\).

b.3 Prove that \( m^*(P_n) = m^*(P) \) for each \( n \geq 1 \), where \( m^* \) is the Lebesgue outer measure.

c) Conclude that \( P \) is not (Lebesgue) measurable. Is \( P \) countable?

2) Let \( E \subset \mathbb{R} \) be Lebesgue measurable.

a.1 Prove that if \( E \subset P \) where \( P \) is the set constructed in Problem 1), then \( m(E) = 0 \).

a.2 Assume that \( m(E) > 0 \). Prove that there exists \( S \subset E \) with \( S \) non (Lebesgue) measurable. [Hint: First prove that the statement will follow if it can be proved for every subset \( E \) of \((0, 1)\).]

3) Solve the following problems from the textbook. 2.3 c–e; 2.6, 2.15, 2.17, 2.18. 2.19. 2.20.