A proper subset of the problems will be selected for grading.

1) Let $X = Y = \mathbb{N}$, $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{P}(\mathbb{N})$, $\mu = \nu$ is the counting measure on $\mathbb{N}$. Define $f(m,n) = 1$ if $m = n$, $f(m,n) = -1$ if $m = n + 1$, and $f(m,n) = 0$ otherwise. Prove that $\int \int |f(m,n)| d(\mu \times \nu)(m,n) = \infty$, and $\int_\mathbb{N}(\int_\mathbb{N} f^nd\mu(m))d\nu(n)$ and $\int_\mathbb{N}(\int_\mathbb{N} f_md\nu(n))d\mu(n)$ exist and are unequal.

2) Let $X = Y = [0,1]$, $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{B}([0,1])$, $\mu = m$ is the Lebesgue measure on $[0,1]$, and $\nu$ is the counting measure on $[0,1]$. If $D = \{(x,x), x \in [0,1]\}$, then $\int \int 1_D d\mu d\nu$, $\int \int 1_D d\nu d\mu$, and $\int \int 1_D d(\mu \times \nu)$ are all inequal.

3) Let $(X, \mathcal{M}, \mu)$ be a $\sigma$–finite measure space and $f \in L_1^1(\mu)$, $f \geq 0$, $\mu$ – a.e. Let

$$G_f = \{(x,y) \in X \times [0,\infty] : y \leq f(x)\}.$$ 

Prove that $G_f$ is $\mathcal{M} \times \mathcal{B}([0,\infty))$–measurable and that $\mu \times m(G_f) = \int f d\mu$.

4) Let $E, F \subset \mathbb{R}$ be measurable, and assume that $f$ is a measurable function on $E \times F$. Prove that $f = 0$, a.e. on $E \times F$ if and only if for almost every $x \in E$, we have $f_x(y) = 0$ for a.e. $y \in F$.

5) Let $([0,\infty), \mathcal{B}([0,\infty)), m)$ be the Borel measure space on $[0,\infty)$. Suppose that $f \in L_1^1([0,\infty))$, $f \geq 0$, $m$ – a.e. such that $\int_0^\infty f dm > 0$, and for $s \geq 0$ let $g(s) = \int_{\{t \leq s\}} f(t) dm(t)$. Prove that $g \notin L^1$. 
