2.3 Limit rules and examples

**Theorem 2.3.1.** If \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist, then \( \lim_{x \to a} \left( f(x) + g(x) \right) \), \( \lim_{x \to a} \left( c f(x) \right) \), \( \lim_{x \to a} \left( f(x) - g(x) \right) \), \( \lim_{x \to a} \left( f(x) \cdot g(x) \right) \), exist and

1. \( \lim_{x \to a} \left( f(x) + g(x) \right) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \), \( \lim_{x \to a} c f(x) = c \lim_{x \to a} f(x) \),
2. \( \lim_{x \to a} \left( f(x) - g(x) \right) = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) \), \( \lim_{x \to a} \left( f(x) \cdot g(x) \right) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) \).
3. If in addition, \( \lim_{x \to a} g(x) \neq 0 \), then \( \lim_{x \to a} \frac{f(x)}{g(x)} \) exists and \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \).

**Remark 2.3.1.** If the limits of \( f_1, f_2, \ldots, f_n \) at \( a \) all exist, then so is the limit of

1. \( f_1 + f_2 + \cdots + f_n \) exists and,
   \[
   \lim_{x \to a} (f_1 + f_2 + \cdots + f_n)(x) = \lim_{x \to a} f_1(x) + \lim_{x \to a} f_2(x) + \cdots + \lim_{x \to a} f_n(x);
   \]
2. and \( f_1 f_2 f_3 \ldots f_n \) exists, and
   \[
   \lim_{x \to a} (f_1 f_2 \cdots f_n)(x) = \lim_{x \to a} f_1(x) \lim_{x \to a} f_2(x) \cdots \lim_{x \to a} f_n(x).
   \]

Consequently, if \( f \) is a polynomial, then for each real number \( a \), \( \lim_{x \to a} f(x) \) exists and \( \lim_{x \to a} f(x) = f(a) \). If \( \frac{f(x)}{g(x)} \) is a rational function and \( \lim_{x \to a} g(x) = g(a) \neq 0 \), then the limit of \( \frac{f(x)}{g(x)} \) at \( a \) exists and \( \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)} \).

**Example 2.3.1.** Justify why each of these limits exist and evaluate them: \( \lim_{x \to 2} (2x^2 - 5x + 7|x|) \), \( \lim_{x \to \sqrt[3]{2}} (x^3 + 3)(-\sqrt[3]{2}x^3 + 1) \), \( \lim_{x \to 1} \frac{x^7}{x^2 - 2x + 3} \).

1. If \( n \) is a positive integer, and \( a \) is a non zero number, then \( \lim_{x \to a} \frac{1}{x^n} = \frac{1}{a^n} \).
2. If \( a \) is a real number and \( r \) is any rational number than \( \lim_{x \to a} x^r = a^r \).
3. In particular \( \lim_{n \to \infty} \sqrt[n]{x} = \sqrt[n]{a} \) for all odd integer \( n \) and all real number \( a \).
   Similarly, \( \lim_{n \to \infty} \sqrt[n]{x} = \sqrt[n]{a} \) for all even integer \( n \) and all real number \( a > 0 \).
4. \( \lim_{x \to a} e^x = e^a \) for all real \( a \).
5. \( \lim_{x \to a} \ln x = \ln a \) for all \( a > 0 \).
6. \( \lim_{x \to 0} \sin x = 0 \), \( \lim_{x \to 0} \cos x = 1 \).

**Definition 2.3.1.** A function \( f \) defined on an open interval containing a number \( a \) such that \( \lim_{x \to a} f(x) = f(a) \) is said to be **continuous at** \( a \).
Theorem 2.3.2 (Squeezing theorem). Assume that \( f(x) \leq g(x) \leq h(x) \) for all \( x \) in some open interval about \( a \) except possibly at \( a \) itself. If \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \), then \( \lim_{x \to a} g(x) \) exists and \( \lim_{x \to a} g(x) = L \).

Example 2.3.2. Prove that \( \lim_{x \to 0} \frac{\sin x}{x} = 1 \), and that \( \lim_{x \to 0} \frac{\cos x - 1}{x} = 0 \).

Theorem 2.3.3 (Substitution rule). If \( \lim_{x \to a} f(x) = c \), then \( \lim_{x \to a} g(f(x)) = \lim_{y \to c} g(y) \). In particular, if \( f \) is continuous at \( a \) and \( g \) is continuous at \( f(a) \), then \( g(f(x)) \) is continuous at \( a \) and \( \lim_{x \to a} g(f(x)) = g(f(a)) \).

Evaluate \( \lim_{x \to 0} \cos \frac{2x^2 + x + \pi}{4} \).

2.4 One-sided and infinite limits

Definition 2.4.1. Let \( f \) be defined on some open interval \((c, a)\). A number \( L \) is the limit of \( f(x) \) as \( x \) approaches \( a \) from the left (or the left-hand limit of \( f \) at \( a \)) if for all \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \( a - \delta < x < a \) then \( |f(x) - L| < \epsilon \).

In this case we write \( \lim_{x \to a^-} f(x) = L \),

and we say that the left-hand limit of \( f \) at \( a \) exists, or that \( \lim_{x \to a^-} f(x) \) exists.

Let \( f \) be defined on some open interval \((a, b)\). A number \( L \) is the limit of \( f(x) \) as \( x \) approaches \( a \) from the right (or the right-hand limit of \( f \) at \( a \)) if for all \( \epsilon > 0 \) there is \( \delta > 0 \) such that if \( a < x < a + \delta \) then \( |f(x) - L| < \epsilon \).

In this case we write \( \lim_{x \to a^+} f(x) = L \),

and we say that the right-hand limit of \( f \) at \( a \) exists, or that \( \lim_{x \to a^+} f(x) \) exists.

Example 2.4.1. Evaluate the following limits

1. \( \lim_{x \to 2^-} \frac{x^3 - 4x}{x^2 - 2} \)
2. \( \lim_{x \to 5^+} \frac{|x - 5|}{x - 5} \)
3. \( \lim_{x \to 3} \sqrt{x^2 - 9} \)

Theorem 2.4.1. Let \( f \) be defined on an open interval about \( a \), except possibly at \( a \) itself. Then \( \lim_{x \to a} f(x) \) exists if and only if both one-sided limits, \( \lim_{x \to a^-} f(x) \) and \( \lim_{x \to a^+} f(x) \) exist and \( \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \). In this case, \( \lim_{x \to a} f(x) = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) \).