1. Use the principle of mathematical induction to show that $1 + 2 + \cdots + n = n(n + 1)/2$ for any positive integer $n$.

2. Suppose $n$ is an integer strictly greater than 1. Using the Fundamental Theorem of Arithmetic write

$$n = p_1^{a_1} p_2^{a_2} \cdots p_r^{a_r}$$

where the $p_i$ are distinct primes and the $a_i$ are positive integers. For each prime number $p$ define $v_p(n)$ to be $a_i$ if $p = p_i$ for some $i$. Define $v_p(n) = 0$ otherwise.

Suppose $m$ is another integer strictly greater than 1. Show that, for all primes $p$, we have

$$v_p(nm) = v_p(n) + v_p(m).$$

3. Define $\nu_p(n)$ to be $a_i$ if $p = p_i$ for some $i$. Define $\nu_p(n) = 0$ otherwise.

Suppose $a$ and $b$ are non-zero integers. Show that $a|b$ if and only, for all primes $p$, $\nu_p(a) \leq \nu_p(b)$.

4. Suppose $n$ and $m$ are two positive integers and let $S = \{p_1, \ldots, p_k\}$ be a finite set of primes containing all of the prime factors of $n$ and all the prime factors of $m$. Using the Fundamental Theorem of Arithmetic write

$$n = \prod_{i=1}^{k} p_i^{a_i},$$

$$m = \prod_{i=1}^{k} p_i^{b_i}.$$ 

Set

$$(n, m) = \prod_{i=1}^{k} p_i^{\min(a_i, b_i)}$$

$$[n, m] = \prod_{i=1}^{k} p_i^{\max(a_i, b_i)}.$$ 

(1) Suppose that $x$ is an integer such that $x|n$ and $x|m$. Show that $x|(n, m)$.

(2) Suppose $y$ is an integer such that $n|y$ and $m|y$. Show that $[n, m]|y$.

(3) Show that $(n, m)[n, m] = nm$.

5. Suppose $f : X \to Y$ and $g : Y \to Z$ are functions. Prove the following:

(1) If $f$ and $g$ are one-one, then so is $g \circ f$.

(2) If $f$ and $g$ are onto, so is $g \circ f$.

(3) If $g \circ f$ is one-one, then so is $f$.

(4) If $g \circ f$ is onto, then so is $g$. 

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