Curves: Length, Tangent and Normal Vector, Curvature

A curve is given by a parametrization
\[ \mathbf{r}(t) = (x(t), y(t), z(t)), \quad a \leq t \leq b \]
The velocity is \( \mathbf{v}(t) = \mathbf{r}'(t) \), the speed is \( V(t) = \| \mathbf{v}(t) \| \), the acceleration is \( \mathbf{a}(t) = \mathbf{r}''(t) \).
The length \( L \) of the curve is given by the integral over the speed
\[ L = \int_a^b V(t)\,dt, \]
the arc length is given by \( s(t) = \int_a^t V(u)\,du \) so that \( \frac{ds}{dt} = V(t) \).

The velocity vector \( \mathbf{v} \) is tangential to the curve at the point \( \mathbf{r}(t) \). The unit tangent vector \( \mathbf{T} \) is defined by
\[ \mathbf{T} = \frac{\mathbf{v}}{V}. \]

We want to consider the function \( T(s) \) which gives the unit tangent vector for a point with arc length \( s \), i.e.,
\[ T(s(t)) = \frac{\mathbf{v}(t)}{V(t)}, \quad \mathbf{v}(t) = V(t)T(s(t)) \]

We take the derivative of this equation and obtain for \( \mathbf{a}(t) = \mathbf{v}'(t) \) with the product and chain rule
\[ \mathbf{a}(t) = \frac{V'(t)T(s(t)) + V(t)T'(s(t))s'(t)}{V(t)} \]
\[ \mathbf{a}(t) = \frac{V'(t)T(s(t))}{V(t)} + \frac{V(t)^2T'(s(t))}{V(t)} \]
Note that \( T(s) \cdot T(s) = 1 \) implies by differentiation \( 2T'(s) \cdot T(s) = 0 \). Hence the vector \( T'(s) \) is orthogonal on the tangent vector \( T \). Therefore we have obtained a decomposition \( \mathbf{a} = \mathbf{a}_{\text{par}} + \mathbf{a}_{\text{orth}} \) where \( \mathbf{a}_{\text{par}} \) is parallel to \( \mathbf{v} \) and \( \mathbf{a}_{\text{orth}} \) is orthogonal on \( \mathbf{v} \).

The length of \( T'(s) \) tells us about the change of the tangent vector as we move along the curve with speed 1, we define this as the curvature \( \kappa \):
\[ \kappa := \| T'(s) \| \]
The normal vector \( \mathbf{N} \) is defined as the unit vector in the direction of \( T'(s) \):
\[ \mathbf{N} = \frac{T'(s)}{\| T'(s) \|}. \]

We therefore have with unit vectors \( \mathbf{T}, \mathbf{N} \) the decomposition
\[ \mathbf{a} = V'T + V^2\kappa N \]
which tells us that the acceleration vector is decomposed into
- a component parallel to the curve with size \( V'(t) \), i.e., the change of speed
- a component orthogonal to the curve with size \( V^2\kappa \), as consequence of the curvature

Recall the case of motion with constant speed \( V \) around a circle \( R \). In this case we obtained an acceleration of size \( V^2\kappa \) with the curvature \( \kappa = 1/R \).

We can find the decomposition \( \mathbf{a} = \mathbf{a}_{\text{par}} + \mathbf{a}_{\text{orth}} \) (where \( \mathbf{a}_{\text{par}} \) is parallel to \( \mathbf{v} \) and \( \mathbf{a}_{\text{orth}} \) is orthogonal on \( \mathbf{v} \)) as follows:
\[ \mathbf{a}_{\text{par}} = \mathbf{p}_v a = \frac{a \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}, \quad \mathbf{a}_{\text{orth}} = \mathbf{a} - \mathbf{a}_{\text{par}} \]
We have $a_{par} = a_T T$ with
\[
a_T = V' = \frac{a \cdot v}{\|v\|}. \tag{4}
\]
We have $a_{orth} = a_N N$ with
\[
a_N = \sqrt{\|a\|^2 - a_T^2} = \frac{\|v \times a\|}{\|v\|}. \tag{5}
\]
The curvature $\kappa$ can then be computed as
\[
\kappa = \frac{a_N}{V^2} = \frac{\|v \times a\|}{V^3}. \tag{6}
\]
The binormal vector $B = T \times N$ is a unit vector which is orthogonal on $v(t)$ and $a(t)$. Hence we can compute it as
\[
B = \frac{v \times a}{\|v \times a\|}
\]

For computing $a_T, a_N, \kappa, T, N$ you should
- find the vectors $v, a$
- find $v \cdot v, v \cdot a, a \cdot a$ from which you get $a_T, a_N, \kappa$ by (4), (5), (6)
- find $T$ using (1), find $N$ using (3) and $N = a_{orth}/\|a_{orth}\|

If you only need $a_T(t_0), a_N(t_0), \kappa(t_0), T(t_0), N(t_0)$ for a given number $t_0$: First compute the two vectors $v(t_0), a(t_0)$. These vectors just contain numbers (without any $t$), and you can do all computations using these two vectors. That’s how you should solve problem 3 below.

**Problem 1**

Let $r(t) = (3t, 4t^{3/2}, -3t^2)$ for $1 \leq t \leq 3$. Find the length of the curve.

**Problem 2**

Let $r(t) = (t^2, t, -t)$. Find the curvature $\kappa(t)$.

**Problem 3**

Let $r(t) = (t, t^2, t^3/3)$. For $t_0 = 1$ compute $N$ and $\kappa$. 