Problem 1: Let $X$ be a normed linear space and let $T$ be a linear functional on $X$. Define the norm

$$
\|T\|_* = \inf\{M : |T(f)| \leq M\|f\| \text{ for all } f \in X\}.
$$

Show that

$$
\|T\|_* = \sup\{T(f) : f \in X, \|f\| \leq 1\}.
$$

Solution: Define $M_* = \sup\{T(f) : \|f\| \leq 1\}$. It follows by restricting the set the supremum is taken over, that

$$
\sup\{T(f) : \|f\| = 1\} \leq M_*.
$$

Therefore for any $f \in X$, since $f/\|f\|$ has norm 1, we conclude that

$$
T\left(\frac{f}{\|f\|}\right) \leq M_*,
$$

which in turn implies that $T(f) \leq M_*\|f\|$. Taking $f \to -f$ and using linearity of $T$ we can conclude, in fact, that $|T(f)| \leq M_*\|f\|$. From the definition of $\|T\|_*$ it follows that

$$
\|T\|_* \leq M_*.
$$

On the other hand, the definition of $\|T\|_*$ also implies that for any $f \in X$, $T(f) \leq \|T\|_*\|f\|$. Whereby, restricting to $f$ such that $\|f\| \leq 1$, we find that $T(f) \leq \|T\|_*$. Taking the sup over all such $f$ with $\|f\| \leq 1$ gives

$$
M_* \leq \|T\|_*.
$$

Therefore we conclude that $M_* = \|T\|_*$. ■

Problem 2: Let $X$ be a normed linear space. Then the collection of bounded linear functionals on $X$ is a linear space on which $\|\cdot\|_*$ is a norm. This normed linear space is called the dual space of $X$ and denoted by $X^*$.

Solution: If $T_1$ and $T_2$ are two continuous linear functionals on $X$ then we can define for any $\alpha, \beta \in \mathbb{R}$, a new continuous linear functional $\alpha T_1 + \beta T_2$, defined for each $f$ in $X$ by

$$
(\alpha T_1 + \beta T_2)(f) = \alpha T_1(f) + \beta T_2(f).
$$
\(\alpha T_1 + \beta T_2\) is clearly continuous and linear, being the sum of two such functions. Since the zero function is also continuous and linear, we conclude that \(X^*\) is a linear space.

Next we check that \(\| \cdot \|_*\) is a norm on this space. Clear \(\| T \|_*\) is non-negative, being the infimum of \(M \geq 0\) such that \(|T(f)| \leq M\| f \|\). If \(T = 0\), then using the characterization of \(\| \cdot \|_*\) defined in the previous problem \(\| T \|_* = 0\). Similarly if \(\| T \|_* = 0\), then \(|T(f)| \leq \| T \|_*\| f \| = 0\) and therefore \(T(f) = 0\) for all \(f\), meaning \(T = 0\). Note, next, by the linearity of \(T\) and the fact that \(\| f \| = \| - f \|, \| - T \|_* = \sup \{-T(f) : \| f \| \leq 1\} = \sup \{T(f) : \| - f \| \leq 1\} = \| T \|_*\).

Therefore for any \(\alpha \in \mathbb{R}\), \(\| \alpha T \|_* = \| \| \alpha \| T \|_*\). We conclude positive homogeneity by

\[
\| \alpha T \|_* = \| \alpha \| T \|_* = \sup \{ |\alpha| T(f) : \| f \| \leq 1\} = |\alpha| \sup \{ T(f) : \| f \| \leq 1\} = |\alpha| \| T \|_*.
\]

The triangle inequality then follows simply from that fact that

\[
\| T_1 + T_2 \|_* = \sup \{T_1(f) + T_2(f) : \| f \| \leq 1\}
\leq \sup \{T_1(f) : \| f \| \leq 1\} + \sup \{T_2(f) : \| f \| \leq 1\}
\leq \| T_1 \| + \| T_2 \|.
\]

**Problem 13:** Fix real numbers \(\alpha\) and \(\beta\). For each natural number \(n\) consider the step function \(f_n\) defined on \(I = [0, 1]\) by

\[
f_n(x) = (1 - (-1)^k)\frac{\alpha}{2} + (1 + (-1)^k)\frac{\beta}{2} \text{ for } \frac{k}{2^n} \leq x \leq \frac{(k + 1)}{2^n}, 0 \leq k < 2^n - 1.
\]

For \(1 < p < \infty\), show that \(\{f_n\}\) converges weakly in \(L^p(I)\) to the constant function that takes the value \((\alpha + \beta)/2\). For \(\alpha \neq \beta\), show that no subsequence of \(\{f_n\}\) converges strongly in \(L^p(I)\).

**Solution:** Suppose \(g\) is continuous, and define \(g_k^n = \int_{k/2^n}^{(k+1)/2^n} g\), then we see that

\[
\int_0^1 f_n g - \frac{\alpha + \beta}{2} \int_0^1 g = \frac{\alpha - \beta}{2} \sum_{k=0}^{2^n-1} (-1)^{k+1} g_k^n.
\]

We see by change of variables that

\[
g_k^n = 2^{-(n-1)} \int_0^{1/2^n} g \left( \frac{k + 2x}{2^n} \right) dx.
\]

Therefore splitting the sum into even and odd terms,

\[
\sum_{k=0}^{2^n-1} (-1)^{k+1} g_k^n = -\int_0^{1/2^n} \left( \sum_{k=0}^{(2^n-1)/2} g \left( \frac{k + x}{2^n} \right) 2^{-(n-1)} - \sum_{k=0}^{(2^n-1)/2} g \left( \frac{k + x}{2^n - 1/2} + 1/2^n \right) 2^{-(n-1)} \right) dx.
\]
Since $g$ is continuous, then for each $x \in [0, 1]$ the following Riemann sums converge
\[
\sum_{k=0}^{2^{(n-1)-1}} g \left( \frac{k + x}{2^{n}} \right) 2^{-(n-1)} \to \int_{0}^{1} g(y) \, dy
\]
and
\[
\sum_{k=0}^{2^{(n-1)-1}} g \left( \frac{k + x}{2^{n}} + \frac{1}{2^{n}} \right) 2^{-(n-1)} \to \int_{0}^{1} g(y) \, dy.
\]
Moreover by the properties of the Riemann integral these sums converge uniformly in $x \in [0, 1]$. Therefore we conclude that
\[
\sum_{k=0}^{2^{n-1}} (-1)^{k+1} g_k^n \to \int_{0}^{1/2} \left( \int_{0}^{1} g(y) \, dy - \int_{0}^{1} g(y) \, dy \right) \, dx = 0.
\]
Since continuous functions are dense in $L^q(I)$ when $1 \leq q < \infty$, and $f_n$ is bounded in $L^p$, it is standard to argue the we may replace $g$ with any function in $L^q$. Therefore
\[
\int_{0}^{1} f_n g \to \frac{\alpha + \beta}{2} \int_{0}^{1} g,
\]
for all $g \in L^q(I)$.

We can see that $\{f_n\}$, nor any subsequence can converge strongly in $L^p$ if $\alpha \neq \beta$, since
\[
\int_{0}^{1} |f_n - (\alpha + \beta)/2|^p = 2^n \left( \left( |(\alpha - \beta)/2|^p 2^{-n} + |(\beta - \alpha)/2|^p 2^{-n} \right) = 2^{1-p} |\alpha - \beta|^p > 0. \]

**Problem 16:** Let $E$ be a measurable set, $\{f_n\}$ a sequence in $L^2(E)$ and $f$ belong to $L^2(E)$. Suppose
\[
\lim_{n \to \infty} \int_{E} f_n \cdot f = \lim_{n \to \infty} \int_{E} f_n^2 = \int_{E} f^2.
\]
Show that $\{f_n\}$ converges strongly to $f$ in $L^2(E)$.

**Solution:** Note that we can expand the norm in $L^2(E)$,
\[
\int_{E} (f_n - f)^2 = \int_{E} f^2 + \int_{E} f_n^2 - 2 \int_{E} f_n \cdot f.
\]
By the assumptions of the problem,
\[
\lim_{n \to \infty} \int_{E} (f_n - f)^2 = \int_{E} f^2 + \int_{E} f^2 - 2 \int_{E} f^2 = 0.
\]
**Problem 20:** Let $1 \leq p_1 < p_2 < \infty$, $\{f_n\}$ be a sequence in $L^{p_2}[0,1]$ and $f$ belong to $L^{p_2}[0,1]$. What is the relationship between $\{f_n\} \rightharpoonup f$ in $L^{p_2}[0,1]$ and $\{f_n\} \rightharpoonup f$ in $L^{p_1}[0,1]$?

**Solution:** Let $q_1 = p_1/(p_1 - 1)$ and $q_2 = p_2/(p_2 - 1)$. Then since $1 < q_2 < q_1 < \infty$, $L^{q_1}[0,1] \subseteq L^{q_2}[0,1]$, we have that if $\{f_n\} \rightharpoonup f$ in $L^{p_2}[0,1]$, then $\{f_n\} \rightharpoonup f$ in $L^{p_1}[0,1]$.

On the other hand, since $\{f_n\}$ and $f$ are bounded in $L^{p_2}[0,1]$, we will see that if $\{f_n\} \rightharpoonup f$ in $L^{p_1}[0,1]$ then $\{f_n\} \rightharpoonup f$ in $L^{q_2}[0,1]$. Indeed let $g \in L^{q_2}[0,1]$, and let $\{\varphi_k\}$ be a sequence of continuous functions on $[0,1]$ such that $\{\varphi_k\} \rightharpoonup g$ in $L^{q_2}[0,1]$, then we see that

$$\int_0^1 (f-f_n)g = \int_0^1 (f-f_n)\varphi_k + \int_0^1 (f-f_n)(g-\varphi_k).$$

We see then by Hölder’s inequality that

$$\left| \int_0^1 (f-f_n)g \right| \leq \left| \int_0^1 (f-f_n)\varphi_k \right| + (\|f\|_{p_2} + \|f_n\|_{p_2})\|g-\varphi_k\|_{q_2}.$$ 

We denote $M = \sup_n \|f_n\|_{p_2} + \|f\| < \infty$. Since $\{\varphi_n\} \subseteq L^{q_1}[0,1]$, we take $n \to \infty$ above to conclude

$$\limsup_n \left| \int_0^1 (f-f_n)g \right| \leq M\|g-\varphi_k\|_{q_2}. $$

Since this is true for all $k$ and $\varphi_k \rightharpoonup g$ in $L^{q_2}[0,1]$, we conclude that

$$\lim_n \left| \int_0^1 (f-f_n)g \right| = 0.$$ 

Therefore $\{f_n\} \rightharpoonup f$ in $L^{p_2}[0,1]$.

**Problem 22:** State and prove the Radon-Riesz Theorem in $\ell^2$.

**Solution:** Suppose $x_n \rightharpoonup x$ in $\ell^2$. Then $\{x_n\} \rightharpoonup x$ in $\ell^2$ if and only if $\|x_n\|_{\ell^2} \to \|x\|_{\ell^2}$. The proof of this result holds in any Hilbert space. We adapt the proof to $\ell^2$ for clarity. Since $\ell^2$ is a normed linear space then clearly by the reverse triangle inequality if $x_n \to x$ in $\ell^2$, then $\|x_n\|_{\ell^2} \to \|x\|_{\ell^2}$.

Now if $\|x_n\|_{\ell^2} \to \|x\|_{\ell^2}$, we see that

$$\sum_{i=1}^\infty (x_n^i - x^i)^2 = \sum_{i=1}^\infty (x_n^i)^2 + \sum_{i=1}^\infty (x^i)^2 - 2 \sum_{i=1}^\infty x_n^i x^i.$$ 

Since by assumption of $\{x_n\} \to x$ in $\ell^2$,

$$\sum_{i=1}^\infty x_n^i x^i \to \sum_{i=1}^\infty (x^i)^2,$$

we conclude that

$$\|x_n - x\|_{\ell^2}^2 = \sum_{i=1}^\infty (x_n^i - x^i)^2 \to 0.$$ 

Therefore $x_n \to x$ in $\ell^2$. 

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