(1) [16 pts] Solve the following initial value problems and give their intervals of definition.

(a) \( \frac{dz}{dx} = \frac{\sqrt{1-z^2}}{x^2-1} \), \( z(0) = 0 \).

(b) \( \frac{dy}{dz} = \frac{z^4e^z + 2zy}{z^2} \), \( y(1) = 0 \).

Solution:

(a) This equation is separable. We write it in separated differential form and integrate

\[
\int_0^z \frac{1}{\sqrt{1-r^2}} \, dr = \int_0^x \frac{1}{s^2-1} \, ds.
\]

The left integral is computed by trig substitution,

\[
\int_0^z \frac{1}{\sqrt{1-r^2}} \, dr = \arcsin z.
\]

The integral on the right is computed by partial fractions

\[
\int_0^x \frac{1}{s^2-1} \, ds = \frac{1}{2} \int_0^x \frac{1}{s-1} \, ds - \frac{1}{2} \int_0^x \frac{1}{s+1} \, ds
\]

\[
= \frac{1}{2} \log \left( \frac{|x-1|}{|x+1|} \right).
\]

Putting these together we find that

\[
\arcsin z = \frac{1}{2} \log \left( \frac{|x-1|}{|x+1|} \right),
\]

which has solution

\[
z = \sin \left( \frac{1}{2} \log \left( \frac{|x-1|}{|x+1|} \right) \right).
\]

This equation is undefined at \( x = 1, -1 \). Since the initial point is \( x = 0 \), the interval of definition is \((-1, 1)\).
(b) This equation is linear and nonhomogeneous and can be written in normal form
\[
\frac{dy}{dz} - \frac{2}{z} y = z^2 e^z.
\]
It has an integrating factor
\[e^{-\int \frac{2}{z} \, dz} = z^{-2},\]
and so in integrating factor form we find
\[
\frac{d}{dz} (z^{-2} y) = e^z.
\]
Integrating this from 1 to \( z \), we find
\[
z^{-2} y = \int_{1}^{z} e^s \, ds = e^z - 1.
\]
Therefore the solution is
\[y = z^2 e^z - z^2.
\]
We can see from the differential equation that the coefficient \( 2/z \) is not defined at \( z = 0 \). It follows that the interval of definition is \( (0, \infty) \).

(2) [10 pts] Consider the following initial value problem,
\[
\frac{dy}{dt} = \frac{(t + \sin y)^{2/3}}{t}, \quad y(t_I) = y_I.
\]
Describe the set of values \((t_I, y_I)\), for which a unique solution exists.

Solution: We can see first that the right hand side is not defined when \( t = 0 \). Away from \( t = 0 \), note that
\[
\partial_y (t + \sin y)^{2/3} = \frac{2}{3} (t + \sin y)^{-1/3} \cos y,
\]
is undefined when \( t + \sin y = 0 \) and continuous for all other values. Therefore we will have a unique solution as long as \( t_I \neq 0 \) and \( t_I + \sin y_I \neq 0 \).

(3) [15 pts] Suppose that due to human elimination of natural predators a population of initially 100 rabbits suddenly flourishes and can be approximated as growing according to a logistic model with growth rate \( R(p) \) equal to \( 1.1 - ap \) (weeks\(^{-1}\)), where \( p \) is the population size and \( a \) is some constant. After a very long amount of time, it is found that the population has settled at 1,100. What was the population size after 3 weeks?

Solution: The model for the population \( p(t) \) must satisfy
\[
\frac{dp}{dt} = (1.1 - ap)p, \quad p(0) = 100.
\]
Note that $p = 1/a$ is the stable stationary point which the population will settle to for large $t$. Therefore $a = 1.1/100 = 10^{-3}$. Since this equation is autonomous, the solution must satisfy the following implicit equation

$$
t = \int_{100}^{p} \frac{1}{1.1 - 10^{-3} r} \, dr
$$

$$
= \frac{1}{1.1} \int_{100}^{p} \frac{1}{r} \, dr + \frac{1}{1.1} 10^{-3} \int_{100}^{p} \frac{1}{1.1 - 10^{-3} r} \, dr
$$

$$
= \log p - \log 100 - \log (1.1 - 10^{-3} p) + \log 1
$$

$$
= \log \left( \frac{10p}{1,100 - p} \right).
$$

Therefore

$$
\frac{10p}{1,100 - p} = e^{t}.
$$

This can be solved for $p$ to give

$$
p = \frac{1,100}{1 + 10e^{-t}}.
$$

The population size after 3 weeks is then

$$
p(3) = \frac{1,100}{1 + 10e^{-3}} \approx 734.4.
$$

(4) [12 pts] Consider the following MATLAB code:

```
% yI = 1;
tI = 0; tF = 10;
n=100;

y = zeros(n+1,1);
y(1) = yI;
h = (tF - tI)/n;
t = tI:h:tF;

for i = 1:n
    f = sin(y(i) + t(i)) - exp(-y(i))*(1-y(i));
    y(i+1) = y(i) + h*f;
end
```

(a) What is the initial value problem being approximated?

(b) Which numerical method is being employed?

(c) What is the order of accuracy?
(d) What is the time step size?

Solution:

(a) The initial value problem being approximated is
\[
\frac{dy}{dt} = \sin(y - t) - e^{-y}(1 - y).
\]

(b) The numerical method is forward Euler.

(c) The order of accuracy is 1.

(d) The time step size is \( h = \frac{10 - 0}{100} = 1/10 \).

(5) [15 pts] Give general implicit solutions to each of the following differential equation.

\[
(2xy + 2x^2y^3 + y^3)dx + (1 + 3xy^2)dy = 0
\]

Solution:

\[
\partial_y(2xy + 2x^2y^3 + y^3) = 2x + 6x^2y^2 + 3y^2
\]
\[
\partial_x(1 + 3xy^2) = 3y^2
\]

and so the equation is not exact. We look for an integrating factor \( \mu \) so that
\[
\partial_y(\mu(2xy + 2x^2y^3 + y^3)) = \partial_x(\mu(1 + 3xy^2))
\]

grouping the terms in \( \mu \) we find
\[
(2xy + 2x^2y^3 + y^3)\partial_y\mu - (1 + 3xy^2)\partial_x\mu = -2x(1 + 3xy^2)\mu
\]
The appearance of \( (1 + 3xy^2) \) on the right-hand side indicates that we should take \( \partial_y\mu = 0 \). The equation then reduces to the ODE
\[
\partial_x\mu = 2x\mu.
\]

Which has solution \( \mu = e^{x^2} \). Upon multiplying the differential equation through by \( \mu \) it is now exact. We now seek an \( H(x, y) \) so that
\[
\partial_x H(x, y) = (2xy + 2x^2y^3 + y^3)e^{x^2}
\]
\[
\partial_y H(x, y) = (1 + 3xy^2)e^{x^2}
\]

We integrate the second equations first
\[
H(x, y) = \int (1 + 3xy^2)e^{x^2} \, dy + h(x) = ye^{x^2} + xy^3e^{x^2} + h(x).
\]
Substituting this into the first equation gives
\[
(2xy + 2x^2y^3 + y^3)e^{x^2} = 2xye^{x^2} + y^3e^{x^2} + 2x^2y^3e^{x^2} + h'(x).
\]
Cancelling out the common terms yields \( h' = 0 \), and so \( h(x) = 0 \). We conclude that our general solutions are given by
\[
y(1 + xy^2)e^{x^2} = c.
\]
(6) [12 pts] Consider the differential equation
\[ \frac{dw}{dt} = we^{-w} \sin w. \]

(a) Identify all stationary points and classify them as stable, unstable or semistable. Sketch its phase-line portrait over \(-2\pi/2 \leq u \leq 3\pi/2\).

(b) Suppose that \(w(0) = -\pi/2\), what is the behavior of \(w(t)\) as \(t \to \infty\).

(c) Suppose that \(w(1) = 3\pi/2\), what is the behavior of \(w(t)\) as \(t \to -\infty\). (Note this is \(t \to -\infty\), not \(\infty\)).

Solution:

(a) The stationary points are given when \(we^{-w} \sin w = 0\). Therefore the roots are given by
\[ w = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots. \]

The root at 0 is a double root, so it is semi-stable. For the other roots \(n\pi\) is stable if \(n\) is odd and \(n > 0\) or \(n\) is even and \(n < 0\), and unstable if \(n\) is even and \(n > 0\) or \(n\) is odd and \(n < 0\). The phase line portrait is

```
- + + -
-\pi 0 \pi
unstable semistable stable
```

(b) If \(w(0) = -\pi/2\) from the phase line portrait we can see that \(\lim_{t \to \infty} w(t) = 0\).

(c) If \(w(1) = 3\pi/2\) then \(\lim_{t \to -\infty} w(t) = 2\pi\).

(7) [10 pts] A 5 kg ball initially at rest is dropped in a fluid that resists its motion with a force of \(v^2/30\) newtons \((= \text{kg m/ sec}^2)\), where \(v\) is the downward velocity of the ball. The acceleration due to gravity is \(9.8 \text{ m/ sec}^2\). Write down (but do not solve) an initial value problem that describes the downward velocity \(v\) as a function of time \(t\). What is the terminal velocity of the ball?

Solution: The equation of motion is
\[ \frac{dv}{dt} = 9.8 - \frac{v^2}{5 \times 30}, \quad v(0) = 0. \]

The terminal velocity is just the stationary point, given by
\[ v_\infty = \sqrt{5 \times 30 \times 9.8} \approx 38.34 \text{ m/sec} \]
(8) [10 pts] Suppose that you use a numerical method to solve a differential equation. Initially you run it with $N = 100$ uniformly spaced grid points, but are unhappy with the accuracy. You increase the number of grid points to $N = 400$. If you want this to reduce the global error by a factor of 25, what is the minimal order of accuracy your method should have? (Please use only integer valued orders).

Solution: You increased the number of grid points by a factor of 4. Therefore you reduced the step size $h$ by the same factor. For a method with accuracy of order $n$ this will reduce the global error by $4^n$. In order for a reduction of a factor of 25, we must solve

$$4^n = 25 \implies n = \log(25)/\log(4) \approx 2.32.$$ 

Therefore we must at least use an order 3 method.