(1) Find a general solution to the following differential equation

\[ xD^2y - (1 + x)Dy + y = x^2e^x, \quad D = \frac{d}{dx}, \]

given that \(e^x\) and \((1 + x)\) are solutions to the homogeneous equation.

**Solution 1:** Put the equation in normal form

\[ D^2y - \frac{1 + x}{x}Dy + \frac{1}{x}y = xe^x. \]

We will use variation of coefficients. The particular solution is given by

\[ Y_P(x) = u_1(x)e^x + u_2(x)(1 + x). \]

We impose that

\[
\begin{align*}
   u_1'e^x + u_2'(1 + x) &= 0 \\
   u_1'e^x + u_2' &= xe^x.
\end{align*}
\]

This can be solved to give \(u_1' = (1 + x), \, u_2' = -e^x\), which has solutions

\[ u_1 = \frac{1}{2}x^2 + x, \quad u_2 = -e^x. \]

Therefore a particular solution is

\[ Y_P = (\frac{1}{2}x^2 + x)e^x - (1 + x)e^x = \frac{1}{2}x^2e^x - e^x. \]

Since \(e^x\) is a homogeneous solution, we can exclude it from \(Y_P\). We find our general solution as

\[ Y(t) = c_1e^x + c_2(1 + x) + \frac{1}{2}x^2e^x. \]

**Solution 2:** Put the equation in normal form

\[ D^2y - \frac{1 + x}{x}Dy + \frac{1}{x}y = xe^x. \]

We will use the Green function method. The Green function for this equation is given by

\[
G(x, s) = \frac{\det \begin{pmatrix} e^s & 1 + s \\ e^x & (1 + x) \end{pmatrix}}{\det \begin{pmatrix} e^s & 1 + s \\ e^s & 1 \end{pmatrix}} = \frac{(1 + x)e^s - (1 + s)e^x}{-se^s}.
\]
Therefore a particular solution is

\[ Y_p(x) = \int_0^x \frac{(1 + s)e^x - (1 + x)e^s}{se^s} se^s \, ds \]

\[ = \int_0^x (1 + s)e^x - (1 + x)e^s \, ds \]

\[ = (x + \frac{1}{2}x^2)e^x - (1 + x)e^x \]

\[ = (\frac{1}{2}x^2 - 1)e^x. \]

Since \( e^x \) is a solution to the homogeneous equation, we may exclude it from the particular solution, therefore

\[ Y_p(x) = \frac{1}{2}x^2e^x, \]

and the general solution is then

\[ Y(t) = c_1e^x + c_2(1 + x) + \frac{1}{2}x^2e^x. \]

(2) Find a particular solution to the following differential equation

\[ D^2y - 2Dy + 2y = e^t \sin t, \quad D = \frac{d}{dt}. \]

**Solution 1:** We will use key identity evaluations. The characteristic polynomial is given by

\[ p(z) = z^2 - 2z + 2 = (z - 1)^2 + 1, \]

which has conjugate pair roots \( r = 1 \pm i \). The forcing is in characteristic form with \( d = 1 \) and \( \mu + i\nu = 1 + i \), which is a simple root of \( p(z) \). Therefore we need the derivative of the Key identity evaluation, which gives

\[ L(te^{(1+i)t}) = p'(1 + i)e^{(1+i)t} = 2i e^{(1+i)t}. \]

Dividing both sides by \( 2i \) and taking the imaginary part of the equation gives

\[ L \left( \text{Im} \left( \frac{1}{2i} te^{(1+i)t} \right) \right) = e^t \sin t. \]

Therefore the particular solution is

\[ Y_p(t) = \text{Im} \left( \frac{1}{2i} te^{(1+i)t} \right) = -\frac{1}{2} te^t \cos t. \]

**Solution 2:** We will use undetermined coefficients. The characteristic polynomial is given by

\[ p(z) = z^2 - 2z + 2 = (z - 1)^2 + 1, \]
which has conjugate pair roots $r = 1 \pm i$. The forcing is in characteristic form with $d = 1$ and $\mu + i\nu = 1 + i$, which is a simple root of $p(z)$. Therefore we are looking for a particular solution of the form

$$Y_P(t) = Ate^t \cos t + Bte^t \sin t.$$  

We see that

$$DY_P = (At + Bt + A)e^t \cos t + (Bt - At + B)e^t \sin t$$  

and

$$D^2Y_P = (2Bt + 2A + 2B)e^t \cos t + (-2At + 2B - 2A)e^t \sin t$$

Putting this into the differential equation we find equation

$$LY_P(t) = 2(2Bt + 2A + 2B)e^t \cos t + 2(-2At + 2B - 2A)e^t \sin t$$

$$- 2(At + Bt + A)e^t \cos t - 2(Bt - At + B)e^t \sin t$$

$$+ 2Ate^t \cos t + 2Bte^t \sin t$$

$$= 2Be^t \cos t - 2Ae^t \sin t$$

Therefore we may choose $B = 0$, $A = -\frac{1}{2}$, which gives

$$LY_P(t) = e^t \sin t.$$  

We conclude that a particular solution is

$$Y_P(t) = -\frac{1}{2}te^t \cos t.$$  

**Solution 3:** We will use the Green function method. The characteristic polynomial is given by

$$p(z) = z^2 - 2z + 2 = (z - 1)^2 + 1,$$

which has conjugate pair roots $r = 1 \pm i$. The Green function is then given by

$$g(t) = c_1e^t \cos t + c_2e^t \sin t,$$

with the initial conditions $g(0) = 0, g'(0) = 1$, this gives the system

$$\begin{cases} g(0) = 0 \Rightarrow c_1 \\ g'(0) = 1 \Rightarrow c_1 + c_2, \end{cases}$$

which has solutions $c_1 = 0, c_2 = 1$. Therefore the Green functions is

$$g(t) = e^t \sin t.$$  

It follows that a particular solution is

$$Y_P(t) = \int_0^t e^{t-s} \sin (t-s)e^s \sin s \, ds$$

$$= e^t \sin t \int_0^t \cos s \sin s \, ds - e^t \cos t \int_0^t \sin^2 s \, ds$$
Here we used the trig identity
\[
\sin (t - s) = \sin t \cos s - \cos t \sin s.
\]
The first integral can be calculated by the substitution \( u = \sin s \), giving
\[
\int_0^t \cos s \sin s \, ds = \frac{1}{2} \sin^2 t
\]
The second integral makes use of \( \sin^2 s = \frac{1}{2} - \frac{1}{2} \cos 2t \), giving
\[
\int_0^t \sin^2 s \, ds = \frac{1}{2} \int_0^t 1 - \cos 2s \, ds
\]
\[= \frac{1}{2} t - \frac{1}{4} \sin 2t
\]
\[= \frac{1}{2} t - \frac{1}{2} \sin t \cos t
\]
Combining these we find our particular solution,
\[
Y_p(t) = \frac{1}{2} e^t \sin^3 t + \frac{1}{2} e^t \cos^2 t \sin t - \frac{1}{2} t e^t \cos t
\]
\[= \frac{1}{2} e^t \sin t - \frac{1}{2} t e^t \cos t.
\]
Since \( e^t \sin t \) is just a solution to the homogeneous equation, we can drop the first term and we find a particular solution
\[
Y_p(t) = -\frac{1}{2} t e^t \cos t.
\]