Study Outline for Exam 2

Second Order Linear Differential Equations:

\[ y'' + p(t)y' + q(t)y = r(t) \quad [*] \text{ (Inhomogeneous)} \]
\[ y'' + p(t)y' + q(t)y = 0 \quad [**] \text{ (Homogeneous)} \]

where \( p, q \) and \( g \) are continuous functions on an interval.

1. Existence and Uniqueness of solutions to IVP:

\[ y(t_0) = y_0, y'(t_0) = y'_0. \quad [***] \]

For any \( t_0 \) in the interval on which the coeff fctns are continuous, there is exactly one solution of \([*]\) satisfying the initial conditions \([***]\). The same is true for \([**]\).

2. The set of solutions to \([**]\) is a two-dimensional vector space, meaning that there are two linearly independent (neither is a multiple of the other) solutions so that EVERY solution is a linear combination of those two.

Two solutions \( y_1 \) and \( y_2 \) are linearly independent and so form a basis for the solution set exactly when the Wronskian, which is given by the following formula and enjoys the property that it is either identically zero or never zero:

\[ W = y_1 y'_2 - y'_1 y_2 \]

is NOT zero.

3. Constant Coefficient Homogeneous Equations

\[ ay'' + by' + cy = 0. \]

(a) The characteristic polynomial is

\[ ar^2 + br + c; \]

Its roots determine the solutions.

Distinct Real Roots: \( r_1, r_2 \) give

\[ e^{r_1 t}, e^{r_2 t}. \]

Complex Conjugate Roots: \( \alpha + i \beta \) give

\[ e^{(\alpha t)\cos(\beta t)}, e^{(\alpha t)\sin(\beta t)}. \]

Repeated Real Root: \( r_0 \) gives

\[ e^{r_0 t}, te^{r_0 t}. \]
4. Reduction of Order (or Order Reduction)

If $y_1(t)$ is a solution of $[**]$, then one gets a second linearly independent solution by substituting

$$y_2(t) = u(t)y_1(t)$$

into $[**]$, noting that the result does not depend on $u$, then solving the resulting differential equation for $u'$ and then integrating to get $u$.

5. Inhomogeneous Equations

If the inhomogeneous term $r(t)$ is a sum of functions, then find a particular solution for each summand and then add them together to get a particular solution for the full equation. We have the following methods:

(a) Undetermined Coefficients (only for constant coeff eqns). When the inhomogeneous term is a product of a polynomial, exponential and a sinusoidal. Try the exact same kind of candidate for a solution using unknown coeffs; plug into the diff eqn to determine the coeffs. Don't forget to multiply the 'candidate' by $t^s$, where $s$ is the smallest non-negative integer required to guarantee that no term in the candidate is a solution of the homogeneous eqn.

(b) Green Function (also for const coeff eqns). Select $g(t)$ to be the unique function that solves the homogeneous eqn and satisfies $g(0)=0$, $g'(0)=1$. Then a particular solution of the inhomog. eqn is given by

$$\int_{t_0}^t g(t-s)r(s) \, ds$$

(c) Variation of Parameters (requires eqn to be normalized, but not const coeff). If $y_1$ and $y_2$ are lin ind sols of the homog eqn, then

$$-y_1(t) \int^t y_2(s)g(s)/W(s) \, ds + y_2(t) \int^t y_1(s)g(s)/W(s) \, ds$$

is a sol of the inhomog eqn.

6. Mass-Spring System

Unforced & Damped or Undamped; i.e.,

$$mu'' + ku = 0 \text{ or } mu'' + \gamma u' + ku = 0.$$ 

yielding harmonic motion or a damped oscillation accordingly.

Forced: Resonance and Beats; i.e.,

$$mu'' + ku = R\cos(\omega t), \ \omega_0 = \sqrt{k/m}.$$ 

Resonance occurs for $\omega = \omega_0$; Beats for $\omega$ very close to $\omega_0$.

7. Finally, you may be expected to interpret a Matlab session involving some qualitative analysis of a linear, non-constant coeff 2nd order ODE.