**Theorem:** \( \text{lcm}(a, b) \times \text{gcd}(a, b) = ab \) for any positive integers \( a, b \).

**Proof:** First a

**Lemma:** If \( m > 0 \), \( \text{lcm}(ma, mb) = m \times \text{lcm}(a, b) \).

Since \( \text{lcm}(ma, mb) \) is a multiple of \( ma \), which is a multiple of \( m \), we have \( m \mid \text{lcm}(ma, mb) \).

Let \( mh_1 = \text{lcm}(ma, mb) \), and set \( h_2 = \text{lcm}(a, b) \).

Then \( ma \mid mh_1 \Rightarrow a \mid h_1 \) and \( mb \mid mh_1 \Rightarrow b \mid h_1 \).

That says \( h_1 \) is a common multiple of \( a \) and \( b \); but \( h_2 \) is the least common multiple, so

\[ h_1 \geq h_2. \quad (1) \]

Next, \( a \mid h_2 \Rightarrow am \mid mh_2 \) and \( b \mid h_2 \Rightarrow bm \mid mh_2 \).

Since \( mh_2 \) is a common multiple of \( ma \) and \( mb \), and \( mh_1 = \text{lcm}(ma, mb) \), we have \( mh_2 \geq mh_1 \), i.e.

\[ h_2 \geq h_1. \quad (2) \]

From (1) and (2), \( h_1 = h_2 \).

Therefore, \( \text{lcm}(ma, mb) = mh_1 = mh_2 = m \times \text{lcm}(a, b) \); proving the Lemma.

**Conclusion of Proof of Theorem:**

Let \( g = \text{gcd}(a, b) \). Since \( g \mid a, g \mid b \), let \( a = gc \) and \( b = gd \).

From a result in the text, \( \text{gcd}(c, d) = \text{gcd}(a/g, b/g) = 1 \).

Now we will prove that \( \text{lcm}(c, d) = cd \). \((3)\)

Since \( c \mid \text{lcm}(c, d) \), let \( \text{lcm}(c, d) = kc \).

Since \( d \mid kc \) and \( \text{gcd}(c, d) = 1 \), \( d \mid k \) and so \( dc \leq kc \).

However, \( kc \) is the least common multiple and \( dc \) is a common multiple, so \( kc \leq dc \).

Hence \( kc = dc \), i.e. \( \text{lcm}(c, d) = cd \).

Finally, using the Lemma and (3), we have:

\[ \text{lcm}(a, b) \times \text{gcd}(a, b) = \text{lcm}(gc, gd) \times g = g \times \text{lcm}(c, d) \times g = g \text{gcd}(g) = (gc)(gd) = ab. \]

**QED**