Problem 5

(a)

Clear[p pend];
pend[t_, x0_, y0_] :=
{x[t], y[t]} /. First[NDSolve[{x'[t] == y[t], y'[t] == -Sin[x[t]],
x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 15}]];
Clear[p];
p[t_] = Flatten[Table[pend[t, 0, y0], {y0, 0.5, 2.5, 0.5}], 0];

ParametricPlot[Evaluate[p[t]], {t, 0, 15},
PlotRange -> {{-Pi, Pi}, {-3, 3}}, AspectRatio -> Automatic]

(b)

For the initial conditions $0.5 \leq \theta'(0) \leq 1.5$, the pendulum exhibits simple back and forth motion: the
points where the trajectories cross the horizontal axis correspond to $\theta' = 0$, i.e., the endpoints of the
swing, whereas the points where the pendulum crosses the vertical axis (or its translates by multiples of
$2\pi$) correspond to the downward (stable) equilibrium position of the pendulum. The translates of the
vertical axis by odd multiples of $\pi$ correspond to the upward (unstable) equilibrium position of the
pendulum. When $\theta'(0) = 2$, the pendulum seems to slow down near the unstable equilibrium point at
$\theta = \pi$ and $\theta' = 0$, then stop. Actually, for this value of $\theta'(0)$, the pendulum should have just enough
energy to approach the vertical position but never swing overhead (see parts (c) and (e)). When
$\theta'(0) = 2.5$, the pendulum has enough energy to swing around and around without stopping.

Depending on your Mathematica installation, you might see the solution curve corresponding to $\theta(0)=2$
go through the upward vertical position. This is due to the small inaccuracy of the numerical method. In
fact, inprevious versions of Mathematica, that is what we used to see in this problem.

(c)

The energy of the pendulum is given by $E(t) = (1/2) y(t)^2 + 1 - \cos x(t)$. If we differentiate this we get
$E' = y y' + x' \sin x$. Using the relations $x' = y$ and $y' = -\sin x$, we get $E = -y \sin x + y \sin x = 0$, so the
energy of the pendulum is constant. This fact is known as the principle of conservation of energy. Notice that if $x(0) = 0$ and $y(0) = 2$, then $E(0) = 2$, which is the same as the energy at the unstable
equilibrium point $x = \pi$, $y = 0$. Thus the solution discussed in part (b) with initial velocity 2 does not have
enough energy to pass through $x = \pi$ (if it did it would have to have a positive velocity, and hence $E > 2$).
(d)

ContourPlot[(1/2) y^2 + 1 - Cos[x], {x, -Pi, 3 Pi}, {y, -3, 3}, ContourShading -> False, Axes -> True, Frame -> False, Contours -> Table[(1/2) y^2, {y, 0, 2.5, 0.5}], PlotPoints -> 60, AxesOrigin -> {0, 0}, AspectRatio -> Automatic]

Since we are considering an ideal, undamped pendulum, the energy of the pendulum will be constant. Therefore the trajectories in the phase plane will lie on curves of constant energy, i.e., level curves of the energy function. This explains the similarity between the graphs in parts (a) and (d). (Mathematically the curves in (a) and (d) are exactly the same. The fact that they look slightly different is a consequence of the fact that we have not drawn a full set of solution curves.)

(e)

If \(x = \pi\), then \(E = (1/2) y^2 + 1 - \cos x = (1/2) y^2 + 2 \geq 2\). That is, the energy is at least 2 when the pendulum is upright, and in fact is greater than 2 unless the velocity \(y\) is 0. Thus \(E_0 = 2\). If \(E < 2\), then the pendulum can never reach the upright position, and must swing back and forth. If \(E > 2\), then the pendulum reaches the upright position with a nonzero velocity and swings overhead; it must keep swinging in the same direction because the velocity can never reach 0. When \(E = 2\), the pendulum approaches the upright position, but correspondingly its velocity approaches 0, and it never quite reaches the upright position, nor does its velocity ever quite reach 0, so it never turns around.

If \(x(0) = 0\) and \(y(0) = b\), then \(E = (1/2) b^2 + 1 - \cos 0 = (1/2) b^2\). Thus if \(b < 2\), then \(E < 2\), and the pendulum swings back and forth. If \(b > 2\), then \(E > 2\), and the pendulum swings overhead. And if \(b = 2\), then \(E = 2\), and the pendulum forever approaches the upright position. This is consistent with what we observed in part (b), except see the final comment in that part, which allows for the possibility that the numerical solution for \(b = 2\) behaved like the actual solution for \(b\) slightly larger than 2 ought to, due to numerical errors.

(f)

Again let \(x = \theta\) and \(y = \theta^t\). Then the corresponding first order system is:

\[
x' = y \\
y' = -0.5 y - \sin x
\]
Clear[x, y];
damppend[t_, x0_, y0_] :=
{x[t], y[t]} /. First[NDSolve[{x'[t] == y[t], y'[t] == -0.5 y[t] - Sin[x[t]],
  x[0] == x0, y[0] == y0}, {x[t], y[t]}, {t, 0, 15}]];
dp[t_] = Flatten[Table[damppend[t, 0, y0], {y0, 0, 6, 0.5}], 0];

ParametricPlot[Evaluate[dp[t]], {t, 0, 15}, PlotRange -> {{-1, 10}, {-2, 6}}]

Almost all the trajectories, and in fact all of the ones in the plot, tend toward one of the stable equilibria represented by \( \theta' = 0 \) and \( \theta = 2 \pi \). Physically, the damping force drains the energy from the pendulum, forcing it to settle toward a stable equilibrium position.

Next we want to estimate the value of \( \theta'(0) \) for which the pendulum tends to the unstable equilibrium position \( \theta' = 0, \theta = \pi \). Physically, this corresponds to the pendulum swinging up toward a stationary "upside down" equilibrium position. From the plot above we can see that the value of \( \theta'(0) \) must be between 3 and 3.5.

Clear[x, y, dp];
dp[t_] = Flatten[Table[damppend[t, 0, y0], {y0, 3, 3.5, .1}], 0];

ParametricPlot[Evaluate[dp[t]], {t, 0, 15}, PlotRange -> {{-1, 8}, {-2, 4}}]
Now we see that the value is between 3 and 3.1. We'll try to get one more decimal place of accuracy.

```math
Clear[x, y, dp];
dp[t_] = Flatten@Table[dampend[t, 0, y0], {y0, 3, 3.1, .01}], 0];
ParametricPlot[Evaluate[dp[t]], {t, 0, 15}, PlotRange -> {{2, 4}, {-1, 1}}]
```

The trajectory which tends to the unstable equilibrium is between the 9th and 10th trajectories in this plot, so the value of \( \theta'(0) \) is between 3.08 and 3.09.

**(g)**

```math
energy[t_, y0_] :=
(1/2) Last[dampend[t, 0, y0]]^2 + 1 - Cos[First[dampend[t, 0, y0]]]
Plot[Evaluate[energy[t, 3]], {t, 0, 15}, AxesLabel -> {time, energy}]
```

Here we see how the energy in the pendulum decays. If we differentiate the energy equation we see that \( E = \theta'' + (\sin \theta) \theta' \), which from the differential equation equals \(-0.5(\theta')^2\). Therefore, the energy
cannot increase, and must decrease unless $\theta' = 0$, in which case the energy momentarily stays the same. Note that $\theta' = 0$ only when the pendulum reaches the extreme point of each swing; at such times the graph above becomes flat for an instant, then resumes its downward trend. Over time, the energy must decrease toward an equilibrium state where the energy is constant. For the energy to be constant, $\theta'$ must be identically zero; that is, the pendulum must be at rest. This is why all the trajectories in part (f) approach one of the rest states where $\theta = 0$ or $2\pi$ and $\theta' = 0$. 