(1) [4pts] A certain college graduate borrows $8000 to buy a car. The lender charges interest at an annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments continuously at a constant annual rate $k$, determine the payment rate $k$ that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

The following numbers might be useful: $\exp(0.3) = 1.3$, $\exp(0.3)/(\exp(0.3) - 1) = 3.86$. To get full credit please make all the computations.

**Solution.**

Let $S(t)$ denote the loan amount after $t$ years. The equation governing $S(t)$ is:

$$\frac{dS}{dt} = 0.1 \cdot S - k$$

with $S(0) = 8000$ and $k$ expressed in $/year$.

The solution to this equation is:

$$S(t) = S(0)e^{rt} - \frac{k}{r}(e^{rt} - 1) = 8000e^{0.1t} - 10k(e^{0.1t} - 1)$$

We know that $S(3) = 0$. Hence:

$$8000e^{0.3} - 10k(e^{0.3} - 1) = 0$$

$$k = \frac{800e^{0.3}}{e^{0.3} - 1} = \frac{800 \cdot 3.86}{3.86 - 1} = \frac{800 \cdot 3.86}{2.86} = \frac{3088}{year} = \frac{3088}{year}$$

The total interest paid the 3-year period is $3k - S_0$, that is:

$$\text{Interest} = 3k - S_0 = 3088 - 8000 = 3088 - 8000 = 1264$$
(2) [6 pts] For the following differential equations: (i) determine the equilibrium points; (ii) classify each equilibrium point; and (iii) sketch the phase line portrait:

(a) \( \frac{dy}{dt} = y(y-1)(y-2), \quad y_0 \geq 0 \)

(b) \( \frac{dy}{dt} = e^{-y} - 1, \quad -\infty < y_0 < \infty \)

(You do not have to solve these equations.)

Solutions.

Problem (a)

(i) The equilibrium points are: 0,1,2.

(ii) We need the sign of \( f(y) \) between the equilibrium points:

\[
\begin{array}{c|c|c|c|c|}
 y & 0 & 1 & 2 & \\
 f(y) & 0 & + & 0 & - & 0 & + \\
\end{array}
\]

Hence: 0 is unstable; 1 is asymptotically stable; and 2 is unstable.

(iii) The phase line portrait is:

Problem (b)

(i) The only equilibrium point is at: \( y=0 \).

(ii) We need the signature of \( f(y) \) around this value:

\[
\begin{array}{c|c|c|c|}
 y & 0 & 0 & - \\
 f(y) & + & 0 & - \\
\end{array}
\]

Hence 0 is an asymptotically stable equilibrium.

(iii) The phase line portrait: