DEPARTMENT OF MATHEMATICS

UNIVERSITY OF MARYLAND

WRITTEN GRADUATE QUALIFYING EXAM

ANALYSIS

August 2006

Instructions

1. Your work on each question will be assigned a grade from 0 to 10. Some problems have multiple parts or ask you to do more than one thing. Be sure to go on to subsequent parts even if there is some part you cannot do. If you are asked to prove a result and then apply it to a given situation you may receive partial credit for a correct application even though you do not give a correct proof.

2. Use a different sheet (or different set of sheets) for each question. Write the problem number and your code number (not your name) on the top of every sheet.

3. Keep scratch work on separate sheets.

4. Unless otherwise stated, you may appeal to a "well-known theorem" in your solution to a problem. However, it is your responsibility to make it clear exactly which theorem you are using and to justify its use.
1. (a) Prove the following version of the Riemann-Lebesgue Lemma: Let \( f \in L^2[\pi, \pi] \). Prove in detail that
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \to 0 \quad \text{as} \quad n \to \infty.
\]
Here \( n \) denotes a positive integer. You may use any of a variety of techniques, but you cannot simply cite another version of the Riemann-Lebesgue Lemma.

(b) Let \( n_k \) be an increasing sequence of positive integers. Show that \( \{ x \mid \lim \inf_{k \to \infty} \sin(n_k x) > 0 \} \) has measure 0.
Notes: You may take it as granted that the above set is measurable.

2. For real \( s \) (only) consider the integral
\[
\int_{-\infty}^{+\infty} \frac{e^{ist}}{t-i} \, dt
\]
(a) Compute the Cauchy Principal Value of the integral (when it exists).
(b) For which values of \( s \) is the integral convergent?

3. Suppose \( (x^p + \frac{1}{x^q}) f \in L^2(0, \infty) \), where \( p > \frac{1}{2} \). Show that \( f \in L^1(0, \infty) \).

4. Let \( D = \{ |z| < 1 \} \) have boundary \( S = \{ |z| = 1 \} \). For \( \zeta \in D \) define
\[
f(z) = \frac{\zeta - \bar{z}^2}{1 - \bar{\zeta} z^2}
\]
(a) Show that \( f(z) \in S \) if and only if \( z \in S \).
(b) Show that \( f \) has at least one fixed point \( \omega \in S \), i.e. \( f(\omega) = \omega \).

5. Let \( f \in L^1(R) \),
\[
F(x) = \int_{R} f(t) \frac{\sin xt}{t} \, dt.
\]
(a) Show that \( F \) is differentiable a.e. and find \( F'(x) \).
(b) Is \( F \) absolutely continuous on closed bounded intervals \([a, b] \)?
6. Let \( \mathcal{F} \) be a family of entire functions. For \( n = 0, \pm 1, \pm 2, \ldots \) define the domains

\[
D_n = \{ n - 2 < \Re(z) < n + 2 \}.
\]

If \( \mathcal{F} \) is normal (i.e. convergence to \( \infty \) is allowed) on each \( D_n \), show that \( \mathcal{F} \) is normal on \( \mathbb{C} \).
a) Since clearly $f(x) e^{-ix} \leq |f(x)| e^{-|x|}$ and since
\[
\int_{-\pi}^{\pi} |f(x)| e^{-|x|} \leq \left( \int_{-\pi}^{\pi} |f(x)|^2 \right)^{1/2} \left( \int_{-\pi}^{\pi} e^{-2|x|} \right)^{1/2} \text{ by Hölder inequality}
\]
which in equal to $\| f(x) \| \leq \int_{-\pi}^{\pi} |x| \, dx = \int_{-\pi}^{\pi} |x| \, dx = 2\pi < \infty$
so $f(x) e^{-ix}$ is bounded by an $L^1$ function. Apply Lebesgue Dominated Convergence Theorem to give us
\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) e^{-in\omega} \, dx = \int_{-\pi}^{\pi} \lim_{n \to \infty} f(x) e^{-in\omega} \, dx \quad \text{sinu} \text{ } f(x) \in L^2(-\pi, \pi)
\]
so $f$ is finite a.e. $\Rightarrow \lim_{n \to \infty} f(x) e^{-in\omega} = 0$ a.e. $\Rightarrow$
\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) e^{-in\omega} \, dx = 0 = \lim_{n \to \infty} \int_{-\pi}^{\pi} f(x) e^{-in\omega} \, dx
\]

b) Clearly $1 \in L^2(-\pi, \pi)$ so
\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} 1 \cdot e^{-in\omega} \, dx = \lim_{n \to \infty} \int_{-\pi}^{\pi} \cos(nx) + i \sin(nx) = 0
\]
Since the limit equals zero the limit of both the real and complex parts must be zero so
\[
\lim_{n \to \infty} \int_{-\pi}^{\pi} \sin(nx) = 0 \quad \text{By Bounded Convergence Theorem}
\]
and since $|\sin(nx)| \leq 1$
\[
0 = \lim_{n \to \infty} \int_{-\pi}^{\pi} |\sin(nx)| = \int_{-\pi}^{\pi} \lim_{n \to \infty} |\sin(nx)| \, dx = \int_{-\pi}^{\pi} \lim_{n \to \infty} |\sin(nx)| \, dx
\]
a positive function $\neq 0$ and only if it is zero a.e. so
\[
\lim_{n \to \infty} |\sin(nx)| \text{ is zero a.e. } \Rightarrow m \{ x \mid \lim_{n \to \infty} \sin(nx) > 0 \} = 0.
\]
\[ \int_{-\infty}^{\infty} \frac{e^{ist}}{t-i} \, dt \quad \text{consider the contour } \gamma \quad \text{, } R>1 \]

Cauchy integral formula tells us the integral should be \( e^{2\pi i} \) around the contour.

So
\[ \int_{-\infty}^{\infty} \frac{e^{ist}}{t-i} \, dt = \lim_{R \to \infty} \int_{\gamma_R} \frac{e^{ist}}{t-i} \, dt = 2\pi \int_{\gamma_R} \frac{e^{ist}}{t-i} \, dt \]

\[ \lim_{R \to \infty} \left| \int_{\gamma_R} \frac{e^{ist}}{t-i} \, dt \right| \leq \lim_{R \to \infty} \int_{\gamma_R} \frac{e^{-sR\sin \theta}}{R^{s+1}} \, d\theta \]

\[ = \int_{0}^{\pi} \lim_{R \to \infty} \frac{e^{-sR\sin \theta}}{R^{s+1}} \, d\theta = 0 \text{ if } s>0 \]
\[ (x^p + \frac{1}{x^p}) f \in L^2(0, \infty) \quad p > \frac{1}{2} \]

\[
\int_0^\infty |f| = \int_0^1 |f| + \int_1^\infty |f|
\]

\[ \text{By Hölder's Inequality,} \]

\[ \int_0^1 |f| \leq \left( \int_0^1 \left| \frac{f}{x} \right|^2 \right)^{1/2} \cdot \left( \int_0^1 \left| x^{2p} \right|^2 \right)^{1/2} \]

\[ \text{Since } f(x^p + \frac{1}{x^p}) \in L^2(0, \infty) \text{ and } \left| f (0 + \frac{1}{x^p}) \right| \leq \left| f (x^p + \frac{1}{x^p}) \right| \]

\[ \text{Clearly } f/x^p \in L^2(0,1) \text{ so } \|f/x^p\|_{L^2(0,1)} = M_1 \]

\[ \int_0^1 |f| \leq M_1 \cdot \sqrt{\frac{1}{2p-1}} \]

\[ \int_1^\infty |f| \leq \left( \int_1^\infty \left| f x^p \right|^2 \right)^{1/2} \cdot \left( \int_1^\infty \left| \frac{x^p}{x} \right|^2 \right)^{1/2} \]

\[ \text{Since } f(x^p + \frac{1}{x^p}) \in L^2(0, \infty) \text{ and } \left| f (x^p + 0) \right| \leq \left| f (x^p + \frac{1}{x^p}) \right| \]

\[ \text{Clearly } f x^p \in L^2(1, \infty) \text{ so } \|f x^p\|_{L^2(1, \infty)} = M_2 \]

\[ \int_1^\infty |f| \leq M_2 \cdot \sqrt{\frac{1}{2p-1}} \]

\[ \int_0^\infty |f| \leq M_1 \sqrt{\frac{1}{2p+1}} + M_2 \cdot \sqrt{\frac{1}{2p-1}} < \infty \text{ so } f \in L^1(0, \infty) \]
a) Let \( \Phi(z) = \frac{e^z - z}{1 - \overline{z}^e} \) and let \( \Psi(z) = z^2 \).

\[ f(z) = (\Phi \circ \Psi)(z) \] suppose \( |f(z)| \leq 1 \)

\( \Psi(z) \) is an automorphism of the disc so if \( |\Phi(z)| \leq 1 \)

then \( |z| \leq 1 \) so if \( |f(z)| \leq 1 \) then \( |\Psi(z)| \leq 1 \)

Now if \( |\Psi(z)| \leq |e^{i\theta}| \leq 1 \) let \( se^{i\theta} \) be any preimage point of \( re^{i\theta} \)

\[ \Psi(se^{i\theta}) = s^2 e^{2i\theta} \text{ and } |\Psi(se^{i\theta})| \leq 1 \]

\[ |s^2 e^{2i\theta}| \leq 1 \Rightarrow |s|^2 \leq 1 \Rightarrow |s| \leq 1 \]

\[ \Rightarrow |se^{i\theta}| \leq 1 \] so any preimage of \( re^{i\theta} \) has norm less than 1.

So \( |f(z)| \leq 1 \Rightarrow |z| \leq 1 \)

b) (By Rouche's Theorem.) Suppose \( f \) has no fixed points

the function \( \Xi \) and \( f(z) \) have no zeros or poles on the boundary of the disc and \( \forall z \in \Xi \implies |z| = 1 \)

\[ |f(z) + z| < |f(z)| + |z| \] because if

\[ |f(z) + z| = |f(z)| + |z| \text{ then } \arg f(z) = \arg (z) \]

since \( \Xi \) sends boundary to boundary as an automorphism of the disc and \( \Psi(\mathbb{D}) \subseteq \mathbb{D} \) also. So if \( \arg f(z) = \arg (z) \) then \( f(z) = z \) which contradicts our assumption that \( f \) has no fixed points.

So since \( |f(z) + z| < |f(z)| + |z| \) on \( \mathbb{D} \) we have in \( \mathbb{D} \)

\[ \text{# of zeros of } f + \text{# of poles of } f = \text{# of zeros of } \Xi + \text{# of poles of } \Xi \]

by Rouché's Theorem.

\[ \text{# of zeros of } z = 1 \text{, # of poles of } z = 0 \text{, # of zeros of } f = 2 \text{ (2nd kind of } z \text{?)} \]

so the equation can't be true & we can't have no fixed points.
I will use the fact that a family of functions is normal iff the family is locally bounded on the domain in question.

Take any compact subset of C, call it K.

Since K is compact, K intersects nontrivially with only finitely many of the \(D_n\)'s, call this finite subset \(D_{n_1}, \ldots, D_{n_k}\).

Then \(D_{n_i} \cap K\) is a compact subset of \(D_{n_i}\) and therefore the family is bounded on \(D_{n_i} \cap K\) with bound \(M_i\).

Therefore on K, we have a bound on the family of \(\max\{M_1, \ldots, M_k\}\).

So the family is bounded on all compact subsets of C and therefore normal.