Unless otherwise stated, you may appeal to a "well-known theorem" in your solution to a problem, but if you do, it is your responsibility to make it clear which theorem you are using and why its use is justified. In problems with multiple parts, be sure to go on to the rest of the problem even if there is some part you cannot do. In working on any part, you may assume the answer to any previous part, even if you have not proved it.

1. Assume that the function \( f \) is of bounded variation on \([0,1]\). For each \( a \in (0,1) \), define \( v(a) \) to be the total variation of the restriction of \( f \) to \([0,a]\). Prove that if \( f \) is absolutely continuous, then \( v \) is also absolutely continuous. Is the conclusion still true if we remove the assumption that \( f \) is absolutely continuous?

2. Evaluate the integral

\[
\int_{-\infty}^{\infty} \frac{e^{ix}}{1 + x^2} \, dx \quad \text{for} \quad t > 0.
\]

Carefully justify convergence.

3. Let \( \{f_n\} \) be a sequence of nonnegative Lebesgue measurable functions on \([0,1]\). Show that \( \{f_n\} \) converges to zero in measure if and only if

\[
\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0.
\]

4. The functions \( f(x) \) and \( g(x) \) are holomorphic in a domain containing the circle \( \gamma \) and its interior. The functions are non vanishing on \( \gamma \). Give a formula for the integral

\[
\int_\gamma \frac{f'(z)g(z)}{f(z)} \, ds
\]

in terms of the values and zeros of the functions.
5. Assume that \( f, g \in L^1(\mathbb{R}, dm) \cap L^\infty(\mathbb{R}, dm) \) where \( m \) denotes the Lebesgue measure on \( \mathbb{R} \), and define the function \( h \) on \( \mathbb{R} \) by
\[
h(x) = \int_{-\infty}^{\infty} f(x+y)g(y)dm(y)
\]
Prove that \( h \) is a continuous function on \( \mathbb{R} \), and that \( \lim_{|x| \to \infty} h(x) = 0 \) for all \( x \) in \( \mathbb{R} \).

6. Statement: a harmonic function \( u(x, y) \) on a simply connected domain \( \Omega \) can be represented in the form \( u(x, y) = \log |f(x)| \) for a holomorphic function \( f(z) \) on \( \Omega \)
   - Prove the statement beginning with the harmonic conjugate
   - Use the statement to show that a harmonic function on \( \Omega \) cannot have an interior minimum or an interior maximum.
First consider the integral \( \int_{\gamma_R} \frac{e^{it\rho}}{1+t^2} \, dt \) where \( \gamma_R \). 

\[ \int_{\gamma_R} \frac{e^{it\rho}}{1+t^2} = \int_{\gamma_R} \frac{e^{it\rho}}{1+t^2} \, dt = \int_{0}^{2\pi} \frac{e^{it\rho}}{1+t^2} \, dt \]

\[ = \frac{2\pi i}{2i} \text{Res}(f; i) \quad \text{and} \quad \text{Res}(f; i) = \frac{e^{it\rho}}{2i} = \frac{e^{-2}}{2i} \]

So, \( 2\pi i \text{Res}(f; i) = \frac{e^{-2}}{2i} \).

\[ \left| \int_{0}^{2\pi} \frac{e^{it\rho}}{1+t^2} \, dt \right| \leq \int_{0}^{2\pi} \left| \frac{e^{it\rho}}{1+t^2} \right| \, dt \leq \int_{0}^{2\pi} \frac{e^{-2\sin^2\theta}}{R^2-1} \, d\theta \leq \int_{0}^{\pi} \frac{R}{R^2-1} \, d\theta = \frac{\pi R}{R^2-1} \]

As \( R \to \infty \), this goes to zero.

\[ \lim_{R \to \infty} \int_{0}^{2\pi} \frac{e^{it\rho}}{1+t^2} \, dt = \lim_{R \to \infty} \left[ \int_{-R}^{R} \frac{e^{it\rho}}{1+t^2} \, dt \right] = \int_{-\infty}^{\infty} \frac{e^{it\rho}}{1+t^2} \, dt \]

\[ \int_{-\infty}^{\infty} \frac{e^{it\rho}}{1+t^2} \, dt = \frac{e^{-t}}{2i} \]
Suppose that \( \bar{s}_n \) converges to \( 2\alpha \) in measure.

\[
\forall \varepsilon > 0 \quad \lim_{n \to \infty} m \{ x \in C_{\alpha} \mid f_n(x) > \varepsilon \} > \frac{2}{3} \varepsilon > 0 \quad \forall \varepsilon > 0
\]

\[
\exists N \text{ s.t. } \forall m > N \quad m \{ x \in C_{\alpha} \mid f_n(x) > \frac{2}{3} \varepsilon \} > \frac{2}{3} \varepsilon
\]

given \( \varepsilon > 0 \)

Let \( A = \{ x \in C_{\alpha} \mid f_n(x) > \frac{2}{3} \varepsilon \} \)

\[
\int \frac{f_n(x)}{1 + f_n(x)} = \int_A \frac{f_n(x)}{1 + f_n(x)} + \int_{A^c} \frac{f_n(x)}{1 + f_n(x)} \\
\leq \int_A \frac{\varepsilon}{2} + \int_{A^c} \frac{1}{2} \\
= \int_C \frac{\varepsilon}{2} + 1 \cdot m(A^c) = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]

\[
\lim_{n \to \infty} \int \frac{f_n(x)}{1 + f_n(x)} < \varepsilon \quad \forall \varepsilon > 0 \quad \text{so} \quad \lim_{n \to \infty} \int \frac{f_n(x)}{1 + f_n(x)} = 0
\]

Now, for the other direction, it is easier to prove the contrapositive. Now suppose

\( \bar{s}_n \) does not converge to \( 2\alpha \) in measure.

\( \exists \varepsilon > 0 \) s.t.

\[
\lim_{n \to \infty} m \{ x \in C_{\alpha} \mid |f_n(x)| \geq \varepsilon \} \neq \alpha.
\]

or

\[
\forall N \in \mathbb{N} \exists m > N \text{ s.t. } m \{ x \in C_{\alpha} \mid |f_n(x)| \geq \varepsilon \} > \frac{2}{3} \varepsilon
\]

Let \( A = \{ x \in C_{\alpha} \mid |f_n(x)| < \varepsilon \} \)

\[
\int \frac{f_n(x)}{1 + f_n(x)} = \int_A \frac{f_n(x)}{1 + f_n(x)} + \int_{A^c} \frac{f_n(x)}{1 + f_n(x)} \geq 0 + \varepsilon^2
\]

Let \( S = \varepsilon^2 \) then \( \forall N \exists m > N \) s.t.

\[
\int \frac{f_n(x)}{1 + f_n(x)} > \varepsilon \quad \text{so} \quad \lim_{n \to \infty} \int \frac{f_n(x)}{1 + f_n(x)} \neq 0.
\]

That was too hard!

Better solution:

\[
0 = \lim_{n \to \infty} \int \frac{f_n(x)}{1 + f_n(x)} \geq \int_{\text{below } A} \frac{\varepsilon}{1 + \varepsilon} = \frac{\text{lim } \varepsilon}{1 + \varepsilon} = \lim_{n \to \infty} \left[ 1 - m(A) \right] = 0 \quad \text{so} \quad \lim_{n \to \infty} m(A) = 2
\]

\[\blacksquare\]
For convenience let's call \( F(z) = \frac{f(z)g(z)}{g(z)} \).

We know from Cauchy's integral formula that
\[
\oint_C F(z) = 2\pi i \sum_{z_i} \text{Res}(F, z_i),
\]
where \( z_i \) are the poles of \( F(z) \).

Notice that a pole occurs in \( F(z) \) exactly when \( f(z) = 0 \) so \( z_1, ..., z_k \) are the zeros of \( f \).

Also notice that since \( g \) is a non-zero \( \alpha(z) g(z) \) will be \( 1 \) for all the zeros on the
region.

This leaves us only to calculate \( \text{Res}(F, z_i) \) for each \( z_i \).

Locally if \( z_i \) is a pole of order \( m_i \) \( f(x) = (z - z_i)^m h(z) \) where \( h(z_i) \neq 0 \).

So \( F(z) = \frac{m_i(z - z_i)^m h(z_i)}{(z - z_i)^m h(z)} g(z) \)

\[
= \frac{m_i h(z_i) + (z - z_i)^m h'(z)}{(z - z_i)^m h(z)} g(z)
\]

which has a simple pole at \( z_i \).

So the Result is \( \frac{m_i h(z_i) + (z - z_i)^m h'(z)}{h(z)} g(z) \bigg|_{z = z_i} = m_i g(z_i) \).

So \( \oint_C F(z) = 2\pi i \sum_{z_i} m_i g(z_i) \) where \( z_1, ..., z_k \) are \( \text{zeros of } f \) of order \( m_1, ..., m_k \).
\[ |h(x_0) - h(x_1)| \leq \int \left| g(y) \left[ f(x_0, y) = f(x_1, y) \right] \right| dy \]

\[ g \in L^1(\mathbb{R}) \quad \text{so} \quad \int g(x) dx \leq M \quad \forall x \in \mathbb{R} \]

\[ \leq \int_{\mathbb{R}} M \left| f(x_0, y) - f(x_1, y) \right| dy \]

\[ = M \int_{\mathbb{R}} |f(x_0 + y) - f(x_1 + y)| dy \]

Since \( F \in L^1(\mathbb{R}) \), we can choose \( n \) large enough such that

\[ \int_{\mathbb{R}} |f(x_0, y) - f(x_1, y)| dy < \epsilon \]

\[ \leq M \left( \int_{-n}^{n} |f(x_0, y) - f(x_1, y)| dy \right) + \epsilon \]

Since \( g \) is continuous on dense in \( L^1([-n, n]) \), \( g \) converges to \( g \) pointwise uniformly.

\[ \int_{-n}^{n} |f(x_0 + y) - f(x_1 + y)| dy < \epsilon \]

So if \( |x_1 - x_0| < \delta \)

\[ |g(x_0 + y) - g(x_1 + y)| < \frac{\epsilon}{2n} \quad \text{so if} \quad |x_1 - x_0| < \delta \]

\[ \text{then} \quad \leq M \left( 3\epsilon \right) \quad \text{so} \quad h \text{ is continuous} \]

b)
If $u(x,y)$ is harmonic, then exists a function $v(x,y)$ such that

$G = u + iv$ is analytic. Note that, if $G$ is analytic then

$F = e^G = e^{u+iv}$ is analytic,

$\log |F| = \log |e^{u+iv}| = \log e^u = u$.

Now, suppose $u$ has a maximum in the interior then $e^u$ has a maximum on the boundary, so $F$ has a maximum on the boundary but by the maximum modulus principle $F$ must be constant. But that implies $u$ is constant.

Similarly if $u$ has an interior minimum, then $F$ has an interior maximum, which implies $F$ has an interior maximum but by maximum modulus principle $F$ is constant $\Rightarrow u$ is constant.